Algebra 4, Homework 3
Due: Sep 11, 2013

(1) Determine the number of generators of the multiplicative group $K^\times$ of a field K of order $q = p^r$.
(2) Prove that every element of $\mathbb{F}_p$ has exactly one p-th root.
(3) Prove that the sum and product of nonzero elements in a finite field K are 0 and -1 respectively, when K has more than 2 elements.
(4) Factor the polynomial $x^{16} - x$ over the fields $\mathbb{F}_2, \mathbb{F}_4$ and $\mathbb{F}_8$.
(5) Given a prime p, describe the integers n such that there exists a field K of order n and an element $\alpha \in K^\times$ whose order is p.
(6) Let $f = x^3 + x + 1$ and $g = x^3 + x^2 + 1$. Show that f and g are irreducible over $\mathbb{F}_2$. Consider the fields K and L obtained by adjoining to $\mathbb{F}_2$ a root of f and g respectively. Are K and L isomorphic? If so, exhibit an explicit isomorphism.
(7) Show that every irreducible polynomial in $\mathbb{F}_p[x]$ divides $x^{p^n} - x$ for some $n \geq 1$.
(8) Let K be a field of order $q = p^r$. If k divides r, show that K contains exactly one field of order p^k.
(9) Show that the subfields of $\mathbb{F}_q$ are totally ordered by inclusion.
(10) Let $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ be the *map*: $\sigma(x) = x^p$ for $x \in \mathbb{F}_q$. Show that σ is a field automorphism. What is its order in the group of all field automorphisms of $\mathbb{F}_q$?