Algebra 4, Homework 2
Due: August 28, 2013

(1) Constructing a square whose area is the same as the area of a circle of unit radius is called *squaring a circle*. Using the fact that π is transcendental, prove that squaring a circle is impossible using a ruler and compass.

(2) Let $\mathcal{C}$ denote the field of constructible real numbers.
   (a) Is $\mathcal{C}$ a finite extension of $\mathbb{Q}$?
   (b) Prove that $\mathcal{C}$ is the smallest subfield of $\mathbb{R}$ with the property that if $a \in \mathcal{C}$ and $a > 0$, then $\sqrt{a} \in \mathcal{C}$.

(3) Let F be a field of characteristic 0. Let $f \in F[x]$ and let f' be its derivative. Let $g \in F[x]$ be an irreducible polynomial which is a common divisor of f and f'. Prove that g^2 divides f.

(4) Is there a field F and a prime p for which $x^p - x$ has a multiple root in F?

(5) Let $f \in F[x]$ be an irreducible polynomial of degree $n \geq 1$. Let $\alpha_1, \ldots, \alpha_n$ be roots of f in an extension K of F. Show that $n \leq [F(\alpha_1, \ldots, \alpha_n) : F] \leq n!$.
   Give examples where both inequalities are realized, for some $n \geq 3$.

(6) Determine
   (i) the number of field homomorphisms $F \to \mathbb{C}$ and
   (ii) the number of field homomorphisms $F \to F$, where

   (a) $F = \mathbb{Q}(\sqrt[3]{2})$,
   (b) $F = \mathbb{Q}(\sqrt[3]{2}, \omega)$, where ω is a primitive third root of unity,
   (c) $F = \mathbb{Q}(i)$