Algebra 4, Homework 1

(1) Let K be a subfield of $\mathbb{C}$ which is not contained in $\mathbb{R}$. Show that K is dense in $\mathbb{C}$ (in the usual metric topology of $\mathbb{C}$).

(2) Let $F \subset R$ where F is a field and R an integral domain. Suppose that R is finite dimensional when viewed as a vector space over F. Show that R is a field.

(3) Find the following:
   (a) $[\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) : \mathbb{Q}]$.
   (b) $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$, where ζ_n is the primitive n-th root of unity.
   (c) $[\mathbb{R} : \mathbb{Q}(\pi)]$.
   (d) $[\mathbb{C}(x) : \mathbb{C}(u)]$, where x is a variable and $u = \frac{x^3}{x+1}$.
   (e) $[\mathbb{Q}(\sqrt{3}, i, \zeta_3) : \mathbb{Q}]$.

(4) Find the irreducible polynomial of $\alpha = \sqrt{3} + \sqrt{5}$ over each of the following fields:
   (a) $\mathbb{Q}$ (b) $\mathbb{Q}(\sqrt{5})$ (c) $\mathbb{Q}(\sqrt{10})$ (d) $\mathbb{Q}(\sqrt{15})$.

(5) Let $\beta = \zeta\sqrt[3]{2}$, where ζ is a primitive third root of unity. Let $K = \mathbb{Q}(\beta)$. Prove that -1 can not be written as a sum of squares in K.

(6) Let ζ_n be the primitive n-th root of unity.
   (a) Find the irreducible polynomial of ζ_4, ζ_6, ζ_8, ζ_9, ζ_{10}, and ζ_{12} over $\mathbb{Q}$.
   (b) Find the irreducible polynomial of ζ_6, ζ_9, and ζ_{12} over $\mathbb{Q}(\zeta_3)$.

(7) Prove or disprove:
   (a) Every algebraic extension is finite.
   (b) If F is an infinite field, then the additive group $(F, +)$ is not cyclic.
   (c) If F is an infinite field, then the multiplicative group $(F \setminus \{0\}, \times)$ is not cyclic.

(8) Let α, β be two complex numbers of degree 3 over $\mathbb{Q}$ and let $K = \mathbb{Q}(\alpha, \beta)$. What are possible values of $[K : \mathbb{Q}]$?

(9) Determine whether or not i is in the field $\mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha + 1 = 0$.

(10) Show that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(11) Let $F \subset K$ and $\alpha \in K$ be algebraic over F. Suppose that $[F(\alpha) : F]$ is odd. Then show that $F(\alpha) = F(\alpha^2)$.

(12) Let $F \subset R \subset K$ with fields F, K and a ring R. If K is algebraic over F, show that R is a field.

(13) Show that the following polynomials are irreducible over $\mathbb{Q}$:
   (a) $x^5 + 4x^4 - 12x^2 + 2x - 14$,
   (b) $x^3 - 3x^2 + 6x + 45$,
   (c) $x^4 - x + 1$,
   (d) $x^4 + x^3 + x^2 + x + 1$.

(14) Let $\alpha = \sqrt[4]{2}$ be the real positive fourth root of 2. Find all fields between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$.