Problem Set 2

1. Let T be a linear operator on a Hermitian space V and let M be the matrix of T with respect to an orthonormal basis of V. Show that:

   (a) M is Hermitian if and only if $< v, Tw >=< Tv, w >$ for $v, w \in W$.

   (b) M is unitary if and only if $< v, w >=< Tv, Tw >$ for $v, w \in W$.

2. Let A, B be positive definite Hermitian matrices. Determine which of the following matrices are positive definite Hermitian: $A^2, A^{-1}, AB, A+B$.

3. Show that the absolute value of the determinant of a unitary matrix is 1.

4. Consider the following functions $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}$:

   (a) $< A, B >= \text{ trace}(A^\star B)$

   (b) $< A, B >= \text{ trace}(\bar{A}B)$

   Determine whether these functions are Hermitian forms and if so determinie their signature.

5. Prove that a real symmetric matrix A is poitive definite if and only if its eigenvalues are positive.

6. Show that the only matrix which is both positive definite, Hermitian and unitary is the identity matrix.

7. Let A be a complex square matrix. Show that ker $A =$(im $A^\star)^\perp$. Here perpendicular is with respect to the standard Hermitian product on $\mathbb{C}^n$.

8. Show that for every complex matrix A there is a unitary matrix P such that $PAP^\star$ is upper triangular.

9. Let A be a Hermitian matrix. Prove that there is a unitary matrix P with determinant 1 such that $PAP^\star$ is diagonal.

10. Determine the type of the quadric $5x^2 + 3xy + 5y^2 - 3x + 2y - 5 = 0$.

11. Let A be a real matrix which is normal and has real eigenvalues. Prove that A is symmetric.

12. Show that a real skew-symmetric matrix is normal.

13. Let A be a normal matrix. Show that

    (a) A is Hermitian if and only if all its eigenvalues are real.

    (b) A is unitary if and only if all its eigenvalues have absolute value 1.

14. Given a complex Hermitian, positive definite matrix A, there exists a Hermitian, positive-definite matrix B such that $B^2 = A$.