Symmetric bilinear forms - Exercises

1. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the form X^tAY.

   (a) $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, (b) $A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right)$.

2. Consider the form given by $A = \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix}\right)$ on $\mathbb{R}^2$. Find the signature of this form. Is it positive definite? If not, find the vectors x such that $x^tAx < 0$.

3. Prove directly that a form represented by the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\mathbb{R}^2$ is positive definite if and only if $a > 0$ and $ad - bc > 0$.

4. (a) Prove that every real square matrix can be written uniquely as a sum of symmetric and skew-symmetric matrices.

   (b) Prove that every bilinear form on a real vector space can be written uniquely as a sum of symmetric and skew-symmetric forms.

5. Let $<,>$ be a symmetric bilinear form on a real vector space V. Define the function $q : V \to \mathbb{R}$ by $q(v) =< v, v >$ for $v \in V$. q is called the *quadratic form* associated to $<,>$. Show that $<,>$ can be recovered from q by expanding $q(v + w)$.

6. Let A be the matrix of a bilinear form with respect to some basis. Prove or disprove:

   (a) the eigenvalues of A are independent of the basis,

   (b) the rank of A is independent of the basis.

7. Let A be a real square matrix which is orthogonal, symmetric and positive definite. Show that A is the identity matrix.

8. Let V be the real vector space of all real polynomials of degree $\leq n$. Define a form on V by

   $$< f, g >= \int_{-1}^{1} f(x)g(x)\, dx.$$

   Show that this is a bilinear form and find an orthonormal basis for V when $n = 1, 2$ or 3.

9. Let V be the vector space of all real $n \times n$ matrices. Define a form on V by

   $$< A, B > \;=\; \text{trace}(A^tB).$$

   Show that this is a bilinear form and find an orthonormal basis.

10. Prove that every **complex** symmetric, nonsingular matrix A has the form $A = P^tP$.

11. Let V be the space of real 2×2 matrices. Consider the bilinear form

    $< A, B >= \text{trace}(AB)$.

    (a) Compute the matrix of the form with respect to the standard basis $\{e_{ij}\}$.

(b) Calculate the signature of this form. Is it a positive definite form?

(c) Find an orthogonal basis for V.

(d) Let W be the subspace of V of trace zero matrices. Determine the signature of the form restricted to W.

12. Let V be the space of real 2×2 matrices. Consider the function:

    $< A, B >= \det(A + B) - \det A - \det B$. Show that this function is a symmetric bilinear form.

    Repeat (a)-(d) of previous problem for this form.

13. Let A be a real square matrix which is positive definite and symmetric. Show that the maximal entries of A are on its diagonal.

14. Consider $V = \mathbb{R}^n$ with dot product. Let T be a linear operator on V. Suppose that the matrix A of T (with respect to some, hence any, basis) is symmetric.

    (a) Prove that V is the orthogonal sum $V = (ker\ T) \oplus (im\ T)$.

    (b) Prove that T is an orthogonal projection into $im\ T$ if and only if $A^2 = A$.

15. Let $<, >$ be a nondegenerate symmetric bilinear form on a real vector space V . Let $W \subset V$ be a subspace, and $W^\perp$ its orthogonal complement. Show that V need not decompose as a direct sum of W and $W^\perp$.

16. Let $\mathbb{B}$ and $\mathbb{B}'$ be two orthonormal bases of a Euclidean space. Show that the change of basis matrix P is orthogonal.

17. Let $<, >$ be a bilinear form a real vector space V and let A represent the form with respect to some basis of V. Show that $<, >$ is degenerate if and only if rank of $A < n$ (where dimension of V is n).

18. If $< v, v >= 0 \Rightarrow v = 0$, then show that the form is nondegenerate. Is the converse true?