SESHADRI CONSTANTS OF SINGULAR LOCI OF LINE ARRANGEMENTS WITH AT MOST TWELVE LINES

MARCIN DUMNICKI, ŁUCJA FARNIK, KRISHNA HANUMANTHU, BRIAN HARBOURNE, PIOTR POKORA, JOAQUIM ROÉ, AND HALSZKA TUTAJ-GASIŃSKA

ABSTRACT. Let $\mathcal{L}$ be an arrangement of d lines in the complex projective plane and let $Z = \mathrm{Sing}(\mathcal{L})$. A question of Pokora asks whether the multipoint Seshadri constant of $\mathcal{O}_{\mathbb{P}^2}(1)$ at Z is always computed by a line of the arrangement containing the largest number of points of Z. We prove that the answer is affirmative for every arrangement of at most twelve lines. The proof uses a Bézout-type covering criterion, elementary restrictions on the intersection lattice, and a detailed analysis of the two extremal combinatorial types that can occur for twelve lines. One type is the Hesse arrangement; it is handled using a net of quartics through its twelve double points. The other type would yield a uniquely determined incidence structure on thirteen lines. We show that this incidence structure is not realizable over any field.

1. INTRODUCTION

Seshadri constants measure the local positivity of nef line bundles. They were introduced by Demailly [5]. Foundational results and systematic treatments include [6, 3, 4, 12]. For a smooth projective surface X, a nef line bundle L, and distinct points $p_1, \ldots, p_r \in X$, the multipoint Seshadri constant is

$$\varepsilon(X, L; p_1, \ldots, p_r) = \inf_C \frac{L \cdot C}{\sum_{i=1}^r \mathrm{mult}_{p_i} C},$$

where the infimum is taken over irreducible reduced curves meeting at least one of the points. Even for $X = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$, exact values are usually difficult to determine. For very general points, the expected value is governed by the Nagata conjecture [13]. Configurations arising from arrangements of curves provide a complementary source of special point sets for which exact computations can sometimes be made. The interaction between line arrangements and algebraic surfaces has a long history; see, for example, [9, 14].

Let

$$\mathcal{L} = \{L_1, \ldots, L_d\} \subset \mathbb{P}^2_{\mathbb{C}}$$

Date: November 4, 2026.

2020 *Mathematics Subject Classification.* Primary 14C20; Secondary 14N20, 52C30.

Key words and phrases. multipoint Seshadri constants, line arrangements, singular loci, Hesse configuration, incidence structures.

be a line arrangement and let

$$Z = \mathrm{Sing}(\mathcal{L}) = \{p \in \mathbb{P}^2 : p \in L_i \cap L_j \text{ for some } i \neq j\}.$$

We write

$$\varepsilon(Z) = \varepsilon\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); Z\big)$$

and set

$$\ell(\mathcal{L}) = \max_{1 \leq i \leq d} \#(L_i \cap Z).$$

Since an arrangement line containing $\ell(\mathcal{L})$ singular points has Seshadri ratio $1/\ell(\mathcal{L})$, one always has

$$\varepsilon(Z) \leq \frac{1}{\ell(\mathcal{L})}.$$

The following question was posed in [15, Question 3.1].

Question 1.1. *Is*

$$\varepsilon(Z) = \frac{1}{\ell(\mathcal{L})}$$

for every complex line arrangement $\mathcal{L}$?

Positive answers were obtained in [15] for several important classes, including star arrangements, Fermat arrangements, and the Klein and Wiman arrangements. The relation with configurational Seshadri constants was further studied in [11], and the question was recently placed in the broader setting of curve configurations on surfaces in [7]. The line-arrangement problem remains open in general.

Our main result settles the first nontrivial range completely.

Theorem 1.2. *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \leq 12$ lines with nonempty singular locus Z. Then*

$$\varepsilon(Z) = \frac{1}{\ell(\mathcal{L})}.$$

The proof has three ingredients. First, an effective curve that passes through Z with a uniform multiplicity gives, by Bézout's theorem, a lower bound for the Seshadri ratios of all curves not contained in it. Second, the identity

$$d - 1 = \sum_{p \in L_i \cap Z} \big(\mathrm{mult}_p \mathcal{L} - 1\big)$$

severely restricts a possible counterexample. These restrictions settle the case $d \leq 11$. For $d = 12$, only two combinatorial types remain. One is the Hesse arrangement. The other would become, after adjoining a line through six double points, a thirteen-line incidence structure with six quadruple and fourteen triple points. We prove that this structure is unique up to relabeling and that it has no realization over any field.

Throughout the paper, arrangements are considered over $\mathbb{C}$, except in Section 5, where the nonrealizability statement is valid over an arbitrary field.

2. A covering criterion and combinatorial restrictions

For an irreducible reduced plane curve D meeting Z, define its Seshadri ratio by

$$\rho_Z(D) = \frac{\deg D}{\sum_{p\in Z} \operatorname{mult}_p D}.$$

If C is an effective, possibly nonreduced, plane curve, put

$$\mu_Z(C) = \min_{p\in Z} \operatorname{mult}_p C.$$

Proposition 2.1 (Covering criterion)**.** *Let C be an effective plane curve of degree k with $\mu_Z(C) \geq t > 0$. If D is an irreducible reduced curve which is not an irreducible component of C, then*

$$\rho_Z(D) \geq \frac{t}{k}.$$

Consequently, if $k/t \leq \ell(\mathcal{L})$ and every irreducible component E of C satisfies $\rho_Z(E) \geq 1/\ell(\mathcal{L})$, then

$$\varepsilon(Z) = \frac{1}{\ell(\mathcal{L})}.$$

Proof. Let $e = \deg D$ and $m_p = \operatorname{mult}_p D$. Since D is not a component of C, Bézout's theorem and the local intersection inequality give

$$ke = C \cdot D \geq \sum_{p\in Z} \operatorname{mult}_p C \operatorname{mult}_p D \geq t \sum_{p\in Z} m_p.$$

Thus $\rho_Z(D) \geq t/k$. Under the additional hypotheses, every irreducible curve has Seshadri ratio at least $1/\ell(\mathcal{L})$, while an arrangement line with $\ell(\mathcal{L})$ singular points gives the opposite inequality. □

The union $L_1 + \cdots + L_d$ has multiplicity at least two at every point of Z. We immediately obtain the following useful observation.

Corollary 2.2. *If $\ell(\mathcal{L}) \geq d/2$, then*

$$\varepsilon(Z) = \frac{1}{\ell(\mathcal{L})}.$$

Proof. Apply Proposition 2.1 to $C = L_1 + \cdots + L_d$, with $k = d$ and $t = 2$. □

For $p \in Z$, denote by $m_p = \operatorname{mult}_p \mathcal{L}$ the number of arrangement lines through p, and let

$$m(\mathcal{L}) = \max_{p\in Z} m_p.$$

For every $L_i \in \mathcal{L}$, counting the other $d-1$ lines according to their intersection point with L_i gives

$$d - 1 = \sum_{p\in L_i \cap Z} (m_p - 1). \tag{2.1}$$

Lemma 2.3. *If $m(\mathcal{L}) \geq \ell(\mathcal{L})$, then the $m(\mathcal{L})$ lines through a point of maximal multiplicity cover Z, and Question 1.1 has an affirmative answer for $\mathcal{L}$.*

Moreover, if $|Z| > 1$ then $m(\mathcal{L}) \leq \ell(\mathcal{L})$.

Proof. If $|Z| = 1$ then the claim is obvious, so assume there is no pencil of lines containing all components of $\mathcal{L}$. Let p be a point of multiplicity $m = m(\mathcal{L})$, and choose an arrangement line N not passing through p. The line N meets the m lines through p in m distinct singular points, so $m \leq \ell(\mathcal{L})$.

Now suppose that $m \geq \ell(\mathcal{L})$, and so in fact $m = \ell(\mathcal{L})$. Every line not through p already contains the m distinct intersections with the lines through p, and hence it contains no additional singular point. It follows that every point of Z lies on one of the m lines through p. Their union has degree $m = \ell(\mathcal{L})$ and multiplicity at least one along Z, so Proposition 2.1 applies. □

In particular, the same covering argument applies whenever the arrangement has a modular point. Thus it recovers the affirmative answer for supersolvable line arrangements; see [10, 14] for the arrangement-theoretic background.

The preceding results give the basic numerical restrictions used below.

Corollary 2.4. *If $\mathcal{L}$ is a counterexample to Question 1.1, then*

$$m(\mathcal{L}) < \ell(\mathcal{L}) < \frac{d}{2}, \qquad \ell(\mathcal{L})\big(m(\mathcal{L}) - 1\big) \geq d - 1. \tag{2.2}$$

Proof. The first two inequalities follow from Corollary 2.2 and Lemma 2.3. The last inequality follows from (2.1), since every line contains at most $\ell(\mathcal{L})$ singular points and every summand is at most $m(\mathcal{L}) - 1$. □

3. Arrangements with at most eleven lines

Proposition 3.1. *Theorem 1.2 holds for $d \leq 11$.*

Proof. Write $\ell = \ell(\mathcal{L})$ and $m = m(\mathcal{L})$. A direct check of the integer solutions of (2.2) shows that for $d \leq 10$ the only possible triple is

$$(d, \ell, m) = (9, 4, 3).$$

In this case equality holds in $\ell(m-1) \geq d-1$. Therefore every line contains exactly four triple points and no other singularities. The union of the nine arrangement lines has multiplicity three along Z, so Proposition 2.1, with $k/t = 9/3 = 3 < 4$, proves the claim.

Let $d = 11$. The only possible pairs are

$$(\ell, m) = (5, 3) \quad \text{and} \quad (5, 4).$$

If $m = 3$, then (2.1) forces every line to contain five triple points. Counting line–triple-point incidences gives

$$55 = 3t_3,$$

a contradiction. Thus this case does not occur.

Assume now that $(\ell, m) = (5, 4)$. If some arrangement line contains no double point, delete it. The remaining ten lines still have multiplicity at least two at every point of Z, because every point on the deleted line had multiplicity at least three. The resulting degree-ten curve covers $2Z$, and all its components contain at most five points of Z. Proposition 2.1 therefore gives $\varepsilon(Z) = 1/5$.

We may consequently assume that every line contains a double point. By (2.1), the list of multiplicities on a line must be one of

$$A = (4, 4, 4, 2), \qquad B = (4, 4, 3, 2, 2), \qquad C = (4, 3, 3, 3, 2).$$

Let a, b, c be the numbers of lines of types A, B, C, respectively. Counting incidences with double and triple points gives

$$a + 2b + c = 2t_2, \qquad b + 3c = 3t_3, \qquad a + b + c = 11. \tag{3.1}$$

The second equation implies $3 \mid b$, while the first and third imply that b is odd. Hence $b = 3$ or $b = 9$.

Suppose first that $b = 3$. Then (3.1) gives $t_2 = 7$. Choose four auxiliary lines covering the seven double points. Together with the arrangement, they form a degree-fifteen curve with multiplicity at least three at every point of Z. Every arrangement component contains at most five points of Z. An auxiliary line not belonging to the arrangement also contains at most five points of Z, since otherwise Bézout with the union of the eleven arrangement lines would give

$$11 \geq 2 \cdot 6.$$

Thus Proposition 2.1 applies with $k/t = 15/3 = 5$.

Finally, suppose that $b = 9$. Then $t_2 = 10$. Partition the ten double points into two sets of five and choose a conic through each set. Adding these two conics to the arrangement again gives a degree-fifteen curve with multiplicity at least three along Z. Its line components have Seshadri ratio at least $1/5$, as above. If an irreducible conic component had ratio smaller than $1/5$, it would contain at least eleven points of Z. Since there are only ten double points, at least one of these eleven points has multiplicity at least three in the arrangement. Bézout would then give

$$22 = 2 \cdot 11 \geq \sum_{p \in Q \cap Z} m_p \geq 10 \cdot 2 + 3 = 23,$$

a contradiction. Proposition 2.1 completes the proof. □

4. The case of twelve lines

By Corollary 2.4, a counterexample with $d = 12$ must satisfy

$$\ell(\mathcal{L}) = 5, \qquad m(\mathcal{L}) = 4.$$

The identity (2.1) shows that the possible multiplicity lists on an arrangement line are

$$(4, 4, 4, 3), \quad (4, 4, 4, 2, 2), \quad (4, 4, 3, 3, 2), \quad (4, 3, 3, 3, 3). \tag{4.1}$$

If a line of type $(4,4,4,3)$ occurs, deleting it leaves the same singular set Z, because all four points retain multiplicity at least two. Some other line still contains five points, so Proposition 3.1 gives $\varepsilon(Z) = 1/5$. If a line of type $(4,3,3,3,3)$ occurs, deletion again leaves the same singular set. Let ℓ' be the maximum number of points of Z on a line of the remaining eleven-line arrangement. Proposition 3.1 gives $\varepsilon(Z) = 1/\ell'$, whereas the deleted line gives $\varepsilon(Z) \le 1/5$. Since $\ell' \le 5$, one obtains $\ell' = 5$ and again $\varepsilon(Z) = 1/5$.

It remains to consider the two line types

$$A = (4,4,4,2,2), \qquad B = (4,4,3,3,2).$$

Let a and b denote their numbers. Counting incidences gives

$$a + b = 12, \qquad 2a + b = 2t_2, \qquad 2b = 3t_3, \qquad 3a + 2b = 4t_4. \tag{4.2}$$

The second equation implies that b is even, the third that $3 \mid b$, and the last, after substituting $a = 12 - b$, that $4 \mid b$. Hence $12 \mid b$, so there are only two cases:

$$(a,b) = (12,0) \quad \text{or} \quad (a,b) = (0,12).$$

4.1. **The Hesse case.** Assume first that $(a,b) = (12,0)$. Then

$$t_2 = 12, \qquad t_3 = 0, \qquad t_4 = 9.$$

The nine quadruple points and the twelve arrangement lines form the classical Hesse $(9_4, 12_3)$ configuration. Indeed, each arrangement line contains three quadruple points, and the twelve lines account for all $\binom{9}{2} = 36$ pairs of quadruple points. Thus the induced incidence structure is the affine plane of order three. Over $\mathbb{C}$ its line realization is projectively equivalent to the standard Hesse arrangement; see [1, 2]. Let ω be a primitive cube root of unity. A convenient equation is

$$xyz \prod_{i,j=0}^{2} \left(x + \omega^i y + \omega^j z\right) = 0. \tag{4.3}$$

Its twelve double points are

$$(1:0:0),\ (0:1:0),\ (0:0:1), \quad \text{and} \quad (1:\alpha:\beta), \qquad \alpha^3 = \beta^3 = 1. \tag{4.4}$$

The nine quadruple points are

$$(0:1:-\gamma), \quad (1:0:-\gamma), \quad (1:-\gamma:0), \qquad \gamma^3 = 1. \tag{4.5}$$

Consider the net of quartics

$$\Lambda = \left\langle x(y^3 - z^3),\ y(x^3 - z^3),\ z(x^3 - y^3) \right\rangle. \tag{4.6}$$

Its base locus is exactly the set in (4.4). Indeed, if $xyz \neq 0$, the three equations imply $x^3 = y^3 = z^3$, while if one coordinate vanishes, their common zeros are precisely the three coordinate vertices. Bertini's theorem [8, Chapter III, Corollary 10.9] shows that a general member is smooth away from the base locus. At each point $(1:\alpha:\beta)$ with $\alpha^3 = \beta^3 = 1$, the differential of $x(y^3 - z^3)$ is nonzero, and at each coordinate vertex one of the three generators has nonzero differential. Hence singularity at any fixed base point is a proper

linear condition on the coefficients of a member of Λ; a general member is therefore smooth at all twelve base points as well. Finally, none of the quadruple points is a base point, so a general member of Λ avoids all nine of them.

Choose a smooth quartic $Q \in \Lambda$ avoiding the quadruple points. It is irreducible and passes simply through the twelve double points. The divisor

$$C = L_1 + \cdots + L_{12} + 2Q$$

has degree 20 and multiplicity four at every point of Z. Every arrangement line has Seshadri ratio $1/5$, while

$$\rho_Z(Q) = \frac{4}{12} = \frac{1}{3}.$$

Proposition 2.1 therefore yields

$$\varepsilon(Z) = \frac{1}{5}.$$

4.2. **The remaining combinatorial type.** Assume now that $(a, b) = (0, 12)$. Then

$$t_2 = 6, \qquad t_3 = 8, \qquad t_4 = 6.$$

If the six double points are not collinear, pair them and add the three joining lines. The resulting degree-fifteen divisor has multiplicity at least three along Z. Any line not belonging to the arrangement contains at most six points of Z by Bézout with the twelve arrangement lines. Equality can occur only when all six points are double points. Since the six double points are not collinear, every component of the covering divisor contains at most five points of Z, and Proposition 2.1 gives $\varepsilon(Z) = 1/5$.

It remains to exclude the possibility that the six double points lie on a line L_{13}. After adjoining L_{13}, the six double points become triple points. We would obtain a configuration of thirteen lines such that each of $L_1, \ldots, L_{12}$ contains two quadruple and three triple points, while L_{13} contains six triple points. Thus the enlarged configuration would have

$$t_3 = 14, \qquad t_4 = 6.$$

We now prove that such a configuration does not exist.

5. A nonrealizable thirteen-line incidence structure

Proposition 5.1. *Up to relabeling, there is a unique incidence structure of thirteen lines with the following properties:*

(i) *each of $L_1, \ldots, L_{12}$ contains two quadruple and three triple points;*
(ii) *L_{13} contains six triple points and no quadruple point;*
(iii) *there are six quadruple and fourteen triple points in total.*

It is described in Table 1.

Proof. Use the six quadruple points as vertices of a graph G, and represent each of $L_1, \ldots, L_{12}$ by the edge joining the two quadruple points on that line. The graph is simple and 4-regular on six vertices. Hence its complement is a perfect matching, and therefore

$$G \cong K_6 \setminus M_0$$

for a perfect matching M_0. This determines the quadruple-point incidences uniquely.

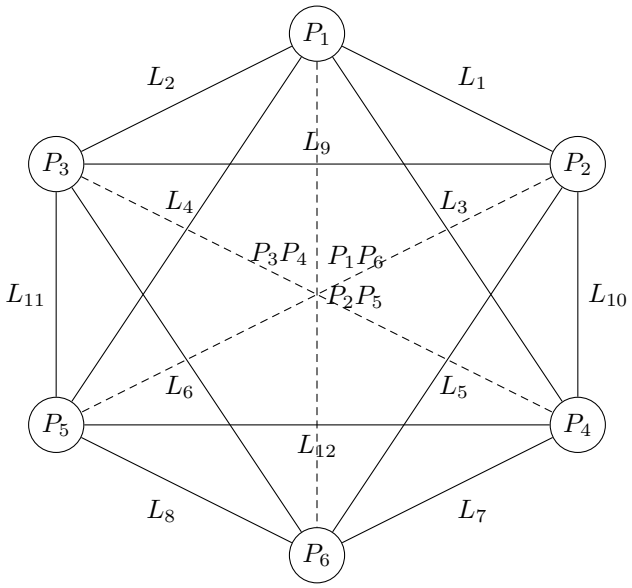


FIGURE 1. The graph $G = K_6 \setminus M_0$, with the edges labeled according to Table 1. The dashed edges form the complementary perfect matching $M_0 = P_1P_6, P_2P_5, P_3P_4$.

Two of the first twelve lines which do not share a quadruple point correspond to disjoint edges of G. A triple point not on L_{13} corresponds to a perfect matching of G. The graph G has exactly eight perfect matchings. Indeed, fix the vertex P_1. It can be matched with any of its four neighbours in G. After such a choice, the subgraph induced by the four remaining vertices is K_4 with exactly one edge removed, and hence has precisely two perfect matchings. Consequently, the total number of perfect matchings of G is

$$4 \cdot 2 = 8.$$

There are exactly eight triple points away from L_{13}, so all these perfect matchings occur. A pair of disjoint edges of G is contained in such a perfect matching unless the two unused vertices form an edge of M_0. The six remaining pairs are disjoint and exhaust the twelve edges of G; they are therefore precisely the six pairs meeting on L_{13}. This determines all incidences and gives Table 1. □

TABLE 1. The unique incidence structure from Proposition 5.1.

point	incident lines	point	incident lines
P_1	L_1, L_2, L_3, L_4	P_{11}	L_2, L_7, L_{13}
P_2	L_1, L_5, L_9, L_{10}	P_{12}	L_2, L_8, L_{10}
P_3	L_2, L_6, L_9, L_{11}	P_{13}	L_3, L_5, L_{11}
P_4	L_3, L_7, L_{10}, L_{12}	P_{14}	L_3, L_6, L_{13}
P_5	L_4, L_8, L_{11}, L_{12}	P_{15}	L_3, L_8, L_9
P_6	L_5, L_6, L_7, L_8	P_{16}	L_4, L_5, L_{13}
P_7	L_1, L_6, L_{12}	P_{17}	L_4, L_6, L_{10}
P_8	L_1, L_7, L_{11}	P_{18}	L_4, L_7, L_9
P_9	L_1, L_8, L_{13}	P_{19}	L_9, L_{12}, L_{13}
P_{10}	L_2, L_5, L_{12}	P_{20}	L_{10}, L_{11}, L_{13}

Theorem 5.2. *The incidence structure in Table 1 is not realizable by lines in $\mathbb{P}^2$ over any field.*

Remark 5.3. The incidence table defines a genuine abstract incidence structure, which is unique up to relabeling. Thus the obstruction is not combinatorial consistency but representability. Over $\mathbb{C}$, its nonrealizability can already be detected by the Hirzebruch-type inequality [16, Remark 11]

$$2t_2 + t_3 \geq d + 3 + \sum_{r\geq 5}(r-4)t_r.$$

Indeed, here $d = 13$, $t_2 = 0$, and $t_3 = 14$, so that $14 < 16$. The coordinate argument below is stronger, since it excludes realizations over an arbitrary field.

Proof. Assume that a realization exists over a field K. The four points P_1, P_9, P_{16}, P_6 are in general position, so after a projective change of coordinates we may take

$$P_1 = (1:0:0), \quad P_9 = (0:0:1),$$
$$P_{16} = (0:1:0), \quad P_6 = (1:1:1).$$

The incidence table then gives

$$L_{13} : x = 0, \quad L_1 : y = 0, \quad L_4 : z = 0, \quad L_5 : x - z = 0, \quad L_8 : x - y = 0.$$

Write

$$P_{14} = (0:t:1), \qquad P_{11} = (0:s:1).$$

The points are distinct, so $s \neq t$. The lines through the prescribed pairs are

$$L_3 : y - tz = 0, \quad L_6 : (1-t)x - y + tz = 0,$$
$$L_2 : y - sz = 0, \quad L_7 : (1-s)x - y + sz = 0.$$

A direct intersection computation yields

$$\begin{aligned} P_2 &= (1:0:1), & P_3 &= (t-s : s(t-1) : t-1),\\ P_4 &= (t-s : t(1-s) : 1-s), & P_{15} &= (t:t:1),\\ P_{18} &= (-1 : s-1 : 0), & P_{12} &= (s:s:1),\\ P_{17} &= (-1 : t-1 : 0). \end{aligned}$$

Consequently, the lines $L_9 = P_2P_3$ and $L_{10} = P_2P_4$ have equations

$$\begin{aligned} L_9 &: s(t-1)x + (s-1)y - s(t-1)z = 0,\\ L_{10} &: t(s-1)x + (t-1)y - t(s-1)z = 0. \end{aligned}$$

The required incidences $P_{15}, P_{18} \in L_9$ and $P_{12}, P_{17} \in L_{10}$ give

$$f_1 = st^2 - st + s - t = 0, \tag{5.1}$$

$$f_2 = s^2 - st - s + 1 = 0, \tag{5.2}$$

$$f_3 = s^2t - st - s + t = 0, \tag{5.3}$$

$$f_4 = t^2 - st - t + 1 = 0. \tag{5.4}$$

Subtracting (5.3) from (5.1) and (5.4) from (5.2) gives

$$(s-t)(st-2) = 0, \qquad (s-t)(s+t-1) = 0.$$

Since $s \neq t$, we obtain $st = 2$ and $s + t = 1$. Substitution into (5.1) gives

$$f_1 = 2t - 2 + s - t = s + t - 2 = -1,$$

a contradiction in every field. □

Proof of Theorem 1.2. Proposition 3.1 treats $d \leq 11$. Section 4 reduces the case $d = 12$ to the Hesse case and the possible collinearity of six double points. The Hesse case was settled using the quartic covering divisor, and Theorem 5.2 excludes the remaining incidence structure. □

6. A counterexample with sixteen lines over $\overline{\mathbb{F}}_7$

Question 1.1 is stated over $\mathbb{C}$, but the definition of $\varepsilon(Z)$ makes sense over any algebraically closed field. The results of Section 2 remain valid in that generality, since their proofs use only Bézout's theorem and the identity (2.1). It is therefore natural to ask whether Question 1.1 has an affirmative answer in positive characteristic. In this section we show that it does not. Over $\overline{\mathbb{F}}_7$ there is an arrangement of sixteen lines, each containing at most seven singular points, together with a line outside the arrangement that contains eight. We then show that this phenomenon is confined to characteristic 7, i.e., the intersection lattice of the arrangement has no realization in any other characteristic.

6.1. **The arrangement.** Let $\mathcal{L} = \{L_1, \dots, L_{16}\}$ be the arrangement in $\mathbb{P}^2_{\overline{\mathbb{F}}_7}$ whose lines are the zero loci of the following linear forms:

(6.1)

$$\begin{array}{llll} L_1\colon y+2z, & L_2\colon y+3z, & L_3\colon y+5z, & L_4\colon x, \\ L_5\colon x+z, & L_6\colon x+3z, & L_7\colon x+y+z, & L_8\colon x+y+6z, \\ L_9\colon x+2y+3z, & L_{10}\colon x+3y+4z, & L_{11}\colon x+4y+z, & L_{12}\colon x+4y+5z, \\ L_{13}\colon x+5y+z, & L_{14}\colon x+5y+6z, & L_{15}\colon x+6y+z, & L_{16}\colon x+6y+2z. \end{array}$$

All sixteen lines are defined over $\mathbb{F}_7$, and hence so is every point of $Z = \operatorname{Sing}(\mathcal{L})$. A direct computation shows that $|Z| = 38$, with

$$t_2 = 16, \qquad t_3 = 12, \qquad t_4 = 8, \qquad t_5 = 2.$$

The two quintuple points are $(1:0:6)$ and $(1:1:6)$. Fourteen of the lines contain seven points of Z, and the remaining two, L_{10} and L_{15}, contain six. Hence

$$\ell(\mathcal{L}) = 7, \qquad m(\mathcal{L}) = 5.$$

Now consider the line

$$M\colon x + 6y + 6z = 0,$$

which does not belong to $\mathcal{L}$. It contains the following eight points of Z, each of which is a double point of $\mathcal{L}$:

point	lines through it	point	lines through it
$(1:2:6)$	L_1, L_5	$(1:6:2)$	L_6, L_{12}
$(1:5:3)$	L_2, L_{10}	$(1:0:1)$	L_8, L_{14}
$(1:3:5)$	L_3, L_{13}	$(1:4:4)$	L_9, L_{11}
$(0:1:6)$	L_4, L_7	$(1:1:0)$	L_{15}, L_{16}

Thus $\rho_Z(M) = 1/8$, and therefore

$$\varepsilon(Z) \le \frac{1}{8} < \frac{1}{7} = \frac{1}{\ell(\mathcal{L})}.$$

So $\mathcal{L}$ is a counterexample to Question 1.1 over $\overline{\mathbb{F}}_7$.

Remark 6.1. The line M is extremal. A line $N \notin \mathcal{L}$ meets each arrangement line in exactly one point, so

$$2\,\#(N \cap Z) \le \sum_{p \in N \cap Z} m_p \le d.$$

Hence N contains at most $d/2$ points of Z, with equality only if N passes through $d/2$ double points and the arrangement lines meet N in pairs. The table above shows that M attains this bound for $d = 16$. Exactly the same situation arose for $d = 12$ in Section 5, where six double points on a line L_{13} would have given $\varepsilon(Z) \le 1/6 < 1/5$. Theorem 5.2 rules that out over every field, whereas for $d = 16$ it occurs in characteristic 7. As it must, the triple $(d, \ell, m) = (16, 7, 5)$ satisfies the restrictions (2.2).

Remark 6.2. This arrangement was found by one of the authors with the help of an AI system (the large language model Claude Opus 5). Its correctness was then checked carefully by the authors.

6.2. **No lift to characteristic zero.** The counterexample is a genuinely positive-characteristic phenomenon, as the following proposition shows. We keep the labeling of (6.1).

Proposition 6.3. *Let K be a field, and let $\mathcal{L}'$ be an arrangement of sixteen lines in $\mathbb{P}^2_K$ with the same intersection lattice as $\mathcal{L}$. Then K has characteristic 7. In particular, $\mathcal{L}$ has no complex realization.*

Proof. No three of L_4, L_8, L_9, L_{12} are concurrent. After a projective change of coordinates we may therefore assume that

$$L_4\colon x=0, \qquad L_8\colon y=0, \qquad L_{12}\colon z=0, \qquad L_9\colon x+y+z=0.$$

The concurrences of $\mathcal{L}$ now determine seven further lines, one at a time. Each line in the following table passes through two intersection points of lines that are already known. In every row these two points are distinct over any field, so they determine the line.

line	passes through	equation
L_{10}	$L_9\cap L_{12}=(1:-1:0),\ L_4\cap L_8=(0:0:1)$	$x+y=0$
L_{13}	$L_4\cap L_{12}=(0:1:0),\ L_8\cap L_9=(1:0:-1)$	$x+z=0$
L_{16}	$L_8\cap L_{12}=(1:0:0),\ L_4\cap L_9=(0:1:-1)$	$y+z=0$
L_1	$L_8\cap L_{12}=(1:0:0),\ L_{10}\cap L_{13}=(1:-1:-1)$	$y-z=0$
L_{11}	$L_1\cap L_4=(0:1:1),\ L_{10}\cap L_{16}=(1:-1:1)$	$2x+y-z=0$
L_5	$L_9\cap L_{12}=(1:-1:0),\ L_{11}\cap L_{13}=(1:-3:-1)$	$x+y-2z=0$
L_6	$L_{13}\cap L_{16}=(1:1:-1),\ L_8\cap L_{11}=(1:0:2)$	$2x-3y-z=0$

Finally, L_4, L_5 and L_6 are concurrent in $\mathcal{L}$; in the coordinates of (6.1) all three pass through $(0:1:0)$. In the present coordinates, $L_4\cap L_5=(0:2:1)$, and substituting this point into the equation of L_6 gives

$$2\cdot 0-3\cdot 2-1=-7.$$

Hence $-7=0$ in K, that is, $\operatorname{char}K=7$. □

7. Further questions

Theorem 1.2 shows that the smallest counterexample to Question 1.1 requires at least thirteen lines. Corollary 2.4 continues to impose

$$m(\mathcal{L})<\ell(\mathcal{L})<\frac{d}{2}, \qquad \ell(\mathcal{L})(m(\mathcal{L})-1)\geq d-1.$$

The covering criterion suggests that the most promising candidates are arrangements for which the singular points of smallest multiplicity cannot be covered efficiently by low-degree curves. It would be interesting to combine these restrictions with databases of realizable rank-three matroids and with interpolation computations for fat point schemes.

Question 7.1. *Does Question 1.1 have an affirmative answer for arrangements of thirteen lines? More generally, is there any complex line arrangement for which the Seshadri constant of the full singular locus is not computed by an arrangement line?*

Acknowledgments

KH is supported by a grant from Infosys Foundation and ANRF MATRICS grant (ANRF/ARGM/2025/002374/MTR). The project originated during the mini-workshop *Seshadri Constants*, held at the Mathematisches Forschungsinstitut Oberwolfach from 10 to 16 November 2019.

AI Declaration

Apart from the example in Section 6.1, which was found by `Claude`, no AI tools were used in preparing this manuscript.

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Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland
Email address: Marcin.Dumnicki@uj.edu.pl

Institute of Mathematics, University of the National Education Commission, Podchorążych 2, 30-084 Kraków, Poland
Email address: Lucja.Farnik@gmail.com

Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India
Email address: krishna@cmi.ac.in

Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588, USA
Email address: brianharbourne@unl.edu

Institute of Mathematics, University of the National Education Commission, Podchorążych 2, 30-084 Kraków, Poland
Email address: piotr.pokora@uken.krakow.pl

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
Email address: jroe@mat.uab.cat

Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland
Email address: halszka.tutaj-gasinska@im.uj.edu.pl