SESHADRI CONSTANTS ON SOME FLAG BUNDLES

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ABSTRACT. Let X be a smooth complex projective curve and let E be a vector bundle on X which is not semistable. We consider a flag bundle $\pi : \mathrm{Fl}(E) \to X$ parametrizing certain flags of fibers of E. The dimensions of the successive quotients of the flags are determined by the ranks of vector bundles appearing in the Harder-Narasimhan filtration of E. We compute the Seshadri constants of nef line bundles on $\mathrm{Fl}(E)$.

1. INTRODUCTION

Seshadri constants were introduced by Demailly [8] in 1990 as a tool to study jet separation of line bundles on complex projective varieties. They have become an important topic in the study of positivity in algebraic geometry.

We quickly recall their definition. Let L be a nef line bundle on a projective variety X. For a point $x \in X$, The *Seshadri constant* of L at x is defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C} \ ,$$

where the infimum is taken over all irreducible and reduced curves $C \subset X$ passing through x. Here $L \cdot C$ denotes the intersection multiplicity, and $\mathrm{mult}_x C$ denotes the multiplicity of the curve C at x. So $\varepsilon(X, L, x)$ depends only the numerical equivalence class of L. If the variety is clear from the context, we denote the Seshadri constant simply by $\varepsilon(L, x)$.

By Seshadri's criterion for ampleness, the line bundle L is ample if and only if $\varepsilon(L, x) > 0$ for all $x \in X$. So Seshadri constants of ample line bundles are positive real numbers.

In order to understand the behaviour of $\varepsilon(L, x)$ as x varies, one defines the following numbers:

$$\varepsilon(L, 1) := \sup_{x \in X} \varepsilon(L, x),$$

and

$$\varepsilon(L) := \inf_{x \in X} \varepsilon(L, x) \, .$$

It is known that the values of both $\varepsilon(L, 1)$ and $\varepsilon(L)$ are achieved at specific points. In fact, $\varepsilon(L, 1) = \varepsilon(L, x)$ for a *very general* point $x \in X$.

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It is easily seen that the following inequalities hold for any point $x \in X$. The dimension of X is denoted by n.

$$0 < \varepsilon(L) \leqslant \varepsilon(L, x) \leqslant \varepsilon(L, 1) \leqslant \sqrt[n]{L^n}.$$

One is interested in computing the precise value of Seshadri constants, or at least in giving some bounds. In general, the larger the Seshadri constant is, the more positive L will be. For example, if L is very ample (in fact, if it is ample and base point free) then $\varepsilon(L, x) \geqslant 1$ for all $x \in X$. On the other hand, it may happen that $\varepsilon(L, x) < 1$ for some x if the ample line bundle L is not base point free.

Note that the above definition of Seshadri constants is meaningful only when the dimension of X is two or more. Seshadri constants have been extensively studied on surfaces and the general picture is understood well in this case; see [2, 9, 4] for a sample. In dimension at least three, a lot is known in specific cases such as abelian varieties and Fano varieties (see [1, 11] for example), but the general picture is not well-understood. For a detailed survey, see [10, Chapter 5] and [3].

In this paper, we compute Seshadri constants of ample line bundles on some flag bundles over complex projective curves. We now recall the construction of the flag bundles that we will study in this paper.

Let E be a vector bundle on a connected smooth projective curve X defined over $\mathbb{C}$. We will assume that E is not semistable. Let

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E$$

be the Harder-Narasimhan filtration of E. For every $1 \leqslant j \leqslant d-1$, let r_j denote the rank of E/E_j. Then we have $r_1 > r_2 > \cdots > r_{d-1}$. We consider the flag bundle

$$\pi : \mathrm{Flag}(r_{k_1}, r_{k_2}, \cdots, r_{k_\gamma}, E) \to X, \tag{1.1}$$

where $1 \leqslant k_1 < k_2 < \cdots < k_\gamma \leqslant d-1$. A point in a fiber over a point $x \in X$ represents successive quotients of the vector space E_x of the following form

$$E_x \to W_{k_1} \to W_{k_2} \to \cdots \to W_{k_\gamma},$$

where $\dim(W_{k_i}) = r_{k_i}$ for $1 \leqslant i \leqslant \gamma$. We will denote this flag bundle (1.1) simply by $\mathrm{Fl}(E)$, since the positive integers $r_{k_1}, r_{k_2}, \cdots, r_{k_\gamma}$ are fixed throughout the article.

The nef cone of the variety $\mathrm{Fl}(E)$ was computed by Biswas and Parameswaran [7]. See Section 2 for a brief description of their results. Using their results, one can explicitly describe the ample line bundles on $\mathrm{Fl}(E)$. In order to compute the Seshadri constants, we will first describe the cone of effective curves of $\mathrm{Fl}(E)$ in Proposition 3.2. This enables us to compute the Seshadri constants.

In [5], the authors computed the Seshadri constants of ample line bundles on Grassmann bundles. Our approach to the flag bundle case is motivated by this paper.

In Theorem 4.2, we give lower and upper bounds for the Seshadri constants of a nef line bundle L on $\mathrm{Fl}(E)$ in terms of the non-negative integers expressing L as a linear combination of the generators of the nef cone of $\mathrm{Fl}(E)$. We then show that the lower bound given in Theorem 4.2 is always achieved at specific points in $\mathrm{Fl}(E)$. We also show that under some additional assumptions on the Harder-Narasimhan filtration of E, the upper bound given in Theorem 4.2 is achieved at general points of $Fl(E)$. We give some examples in Section 4.2.

We work throughout over the field $\mathbb{C}$ of complex numbers. The field of real numbers is denoted by $\mathbb{R}$.

2. Nef cone of $Fl(E)$

In this section we recall the description of the nef cone of $Fl(E)$ given in [7].

Let X be a nonsingular projective variety. The *Néron-Severi group* of X is defined to be the quotient group

$$NS(X) = Div(X)/Num(X),$$

where $Div(X)$ is the group of divisors on X and $Num(X)$ is the subgroup of numerically trivial divisors. Then $NS(X)$ is a finitely generated abelian group.

The *Néron-Severi space* of X is defined to be

$$NS(X)_{\mathbb{R}} := NS(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The cone in $NS(X)_{\mathbb{R}}$ generated by all the real nef divisors is called the *nef cone* of X. The cone generated by all the ample divisors is called the *ample cone* of X. The *pseudo-effective cone* of X is the closed cone in $NS(X)_{\mathbb{R}}$ generated by all the effective classes in $NS(X)_{\mathbb{R}}$. An element of the pseudo-effective cone is called a pseudo-effective divisor.

A nef divisor is a limit of ample divisors and a pseudo-effective divisor is a limit of effective classes. In particular, a nef divisor is pseudo-effective.

We will also be interested in the cone of curves in X. Let $N_1(X)_{\mathbb{R}}$ denote the vector space of real one-cycles on X modulo numerical equivalence. There is a perfect pairing between $NS(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$ given by the intersection pairing on X.

The *closed cone of curves* on X, denoted $\overline{NE}(X)$, is the closure of the cone spanned by all effective one-cycles on X. The nef cone and $\overline{NE}(X)$ are dual to each other under the intersection pairing.

For more details, see [10, Section 1.4.C].

We are interested in these cones for the flag bundle $Fl(E)$. Its nef cone was described by [7].

For every $1 \leqslant i \leqslant \gamma$, let

$$f_i : Gr_{r_{k_i}}(E) \to X$$

be the Grassmannian bundle of rank r_{k_i} quotients of the vector bundle E. Let $\mathcal{L}_i$ be the pullback $f_i^*(\mathcal{L}')$, where $\mathcal{L}'$ is a line bundle on X of degree 1.

Consider the following sequence of maps.

$$\mathrm{Fl}(E) \overset{\Phi}{\hookrightarrow} \prod_{i=1}^{\gamma} \mathrm{Gr}_{r_{k_i}}(E) \hookrightarrow \prod_{i=1}^{\gamma} \mathbb{P}(\wedge^{r_{k_i}} E) \tag{2.1}$$

The first embedding Φ is defined by sending a point

$$Q = (E_x \to W_{k_1} \to W_{k_2} \to \cdots \to W_{k_\gamma}) \in \mathrm{Fl}(E)$$

to the tuple

$$(E_x \to W_{k_1}, E_x \to W_{k_2}, \cdots, E_x \to W_{k_\gamma}) \in \prod_{i=1}^{\gamma} \mathrm{Gr}_{r_{k_i}}(E).$$

The second embedding is a product of the Plücker embeddings.

For every $1 \leqslant i \leqslant \gamma$, define the map

$$\Phi_i := \mathrm{Pr}_i \circ \Phi : \mathrm{Fl}(E) \to \mathrm{Gr}_{r_{k_i}}(E), \tag{2.2}$$

where

$$\mathrm{Pr}_i : \prod_{i=1}^{\gamma} \mathrm{Gr}_{r_{k_i}}(E) \to \mathrm{Gr}_{r_{k_i}}(E)$$

is the i-th projection. Let θ_i be the degree of the quotient E/E_{k_i}. Define

$$\omega_i := \mathcal{O}_{\mathrm{Gr}_{r_{k_i}}(E)}(1) - \theta_i \mathcal{L}_i.$$

Then we have the following results.

Theorem 2.1. [7, Proposition 4.1] *Let $1 \leqslant i \leqslant \gamma$. The divisor ω_i is nef. Further ω_i and $\mathcal{L}_i$ generate the nef cone of $Gr_{r_{k_i}}(E)$.*

Let $\tilde{\omega_i}$ be the pullback $\Phi_i^*(\omega_i)$ and let $\mathcal{L} = \pi^*(\mathcal{L}')$, where $\mathcal{L}'$ is a degree 1 line bundle on X.

Theorem 2.2. [7, Theorem 5.1] *The divisors $\tilde{\omega}_1, \ldots, \tilde{\omega}_\gamma, \mathcal{L}$ generate the nef cone of $Fl(E)$.*

Remark 2.3. The description of the nef cone of $\mathrm{Fl}(E)$ given in [7] is valid in general for any flag bundle over X. We consider only flags with dimensions of successive quotients determined by the Harder-Narasimhan filtration of E since this is needed for later computations.

3. Effective cone of curves on Fl(E)

In this section, we describe some curves in $\mathrm{Fl}(E)$ which are dual to the generators of the nef cone given in Theorem 2.2.

Let $x \in X$ be an arbitrary point. We claim that there exist smooth rational curves $C_1, \ldots, C_\gamma$ in $\mathrm{Fl}(E)$ satisfying the following properties:

(1) Each C_i is contained in the fiber $\pi^{-1}(x) = \mathrm{Fl}(E_x)$.

(2) C_i are lines in the following sense: for every $1 \leqslant i \leqslant \gamma$, the image $\Phi_i(C_i)$ inside the fiber $\mathrm{Gr}_{r_{k_i}}(E_x)$ is a line with respect to the Plücker embedding.

(3) $\Phi_j(C_i)$ is a point for every $j \neq i$.

The construction is as follows. Let n denote the rank of E. Fix a $\mathbb{C}$–basis $\{e_1, e_2, \cdots, e_n\}$ of the fiber E_x. For every $1 \leqslant j \leqslant \gamma$, set $s_{k_j} := n - r_{k_{\gamma-j+1}}$.

Now let $1 \leqslant i \leqslant \gamma$. For every $[t_1 : t_2] \in \mathbb{P}^1$, define a flag of subspaces of E_x

$$E_x \supset V_{k_1} \supset V_{k_2} \supset \cdots \supset V_{k_{\gamma-i+1}}[t_1, t_2] \supset \cdots \supset V_{k_\gamma}, \tag{3.1}$$

as follows:

$$\begin{aligned} V_{k_j} &:= \mathbb{C}\langle e_1, e_2, \cdots, e_{s_{k_j}}\rangle, \text{ for } j < \gamma - i + 1, \\ V_{k_{\gamma-i+1}}[t_1, t_2] &:= \mathbb{C}\langle e_1 + e_2, e_2 + e_3, \cdots, e_{s_{k_{\gamma-i+1}}-1} + e_{s_{k_{\gamma-i+1}}}, t_1 e_{s_{k_{\gamma-i+1}}} + t_2 e_{s_{k_{\gamma-i+1}}+1}\rangle, \\ V_{k_j} &:= \mathbb{C}\langle e_1 + e_2, e_2 + e_3, \cdots, e_{s_{k_j}} + e_{s_{k_j}+1}\rangle, \text{ for } j > \gamma - i + 1. \end{aligned}$$

Note that $\dim(V_{k_j}) = s_{k_j} = n - r_{k_{\gamma-j+1}}$ for all $1 \leqslant j \leqslant \gamma$.

The set of all flags of subspaces of E_x as in (3.1) will be denoted by C_i and it is a subset of $\mathrm{Fl}(E_x)$, and hence of $\mathrm{Fl}(E)$. Equivalently using the quotient notation C_i is the following set of flags of quotient spaces of E_x:

$$E_x \to W_{k_1} \to W_{k_2} \to \cdots \to W_{k_i}[t_1, t_2] \to \cdots \to W_{k_\gamma},$$

where $W_{k_j} = (E_x/V_{k_{\gamma-j+1}})$ for $j \neq i$, and $W_{k_i}[t_1, t_2] = (E_x/V_{k_{\gamma-i+1}}[t_1, t_2])$. Note that $\dim(W_{k_j}) = r_{k_j}$ for every $1 \leqslant j \leqslant \gamma$.

From the construction it follows that $\Phi_j(C_i)$ is the constant point $E_x \to W_{k_j}$ for $j \neq i$. Let $\Phi_i(C_i)$ be the image in $\mathrm{Gr}_{r_{k_i}}(E_x)$. It follows from the construction that the image of $\Phi_i(C_i)$ under the Plücker embedding inside $\mathbb{P}(\wedge^{r_{k_i}} E_x)$ has dimension one and it is defined by linear homogeneous polynomials. Hence the image is a line which gives a variety structure on C_i as a subvariety of $\mathrm{Fl}(E)$.

Remark 3.1. The lines C_i in the above construction can be defined functorially. The part of the subspace $V_{k_{\gamma-i+1}}[t_1, t_2]$ that depends on $[t_1 : t_2]$ is the line

$$\{t_1 e_{s_{k_{\gamma-i+1}}} + t_2 e_{s_{k_{\gamma-i+1}}+1}\} \subseteq \mathbb{C}\langle e_{s_{k_{\gamma-i+1}}}, e_{s_{k_{\gamma-i+1}}+1}\rangle.$$

We have an isomorphism $\mathbb{C}\langle e_{s_{k_{\gamma-i+1}}}, e_{s_{k_{\gamma-i+1}}+1}\rangle \cong \mathbb{C}^2$ by sending

$$e_{s_{k_{\gamma-i+1}}} \mapsto (1, 0), \quad e_{s_{k_{\gamma-i+1}}+1} \mapsto (0, 1).$$

We have the family of lines passing through the origin parameterized by $\mathbb{P}^1$

$$\{t_1 \cdot (1, 0) + t_2 \cdot (0, 1)\} \subseteq \mathbb{C}^2.$$

More explicitly, the family is the following

$$\{((x, y), (t_1, t_2)) : t_1 \cdot y = t_2 \cdot x\} \subseteq \mathbb{C}^2 \times \mathbb{P}^1.$$

This is precisely the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \subseteq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Thus the subspaces $V_{k_{\gamma-i+1}}[t_1, t_2]$ will define the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ of rank $s_{k_{\gamma-i+1}}$ over $\mathbb{P}^1$. We have the following filtration of vector bundles over $\mathbb{P}^1$:

$$\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \supset \mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_1}} \supset \cdots \supset \mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_{\gamma-i+1}}-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \supset \mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_{\gamma-i+2}}} \supset \cdots \supset \mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_\gamma}},$$

where the embedding $\mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_{\gamma-i+2}}} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_{\gamma-i+1}}-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is given by $(\mathrm{id}, 0)$. Since $s_{k_{\gamma-i+2}} < s_{k_{\gamma-i+1}}$, by id we mean the inclusion of $s_{k_{\gamma-i+2}}$ copies of $\mathcal{O}_{\mathbb{P}^1}$ inside $(s_{k_{\gamma-i+1}} - 1)$ copies of $\mathcal{O}_{\mathbb{P}^1}$.

The embedding $\mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_{\gamma-i+1}}-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus s_{k_{\gamma-i}}}$ is the following natural embedding

$$\mathbb{P}^1 \times \big(\mathbb{C}\langle e_1+e_2, e_2+e_3, \cdots, e_{s_{k_{\gamma-i+1}}-1}+e_{s_{k_{\gamma-i+1}}}\rangle\big) \oplus \mathbb{C}\langle t_1 e_{s_{k_{\gamma-i+1}}} + t_2 e_{s_{k_{\gamma-i+1}}+1}\rangle \subseteq \mathbb{P}^1 \times \big(\mathbb{C}\langle e_1, e_2, \cdots, e_{s_{k_{\gamma-i}}}\rangle\big),$$

where $s_{k_{\gamma-i}} > s_{k_{\gamma-i+1}}$. The above filtration will define a unique map

$$\mathbb{P}^1 \to \mathrm{Fl}(E_x)$$

and the image of this map is exactly C_i which was defined above.

We recall that the Harder-Narasimhan filtration of E is given by

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E.$$

The rank of E/E_{k_i} is r_{k_i}. So $E/E_{k_{i+1}} = (E/E_{k_i})/(E_{k_{i+1}}/E_{k_i})$. This gives the following sequence of quotients over X:

$$E \to E/E_{k_1} \to E/E_{k_2} \to \cdots \to E/E_{k_\gamma}. \tag{3.2}$$

This defines a section $s : X \to \mathrm{Fl}(E)$.

Proposition 3.2. *The curves* $C_1, \ldots, C_\gamma, s(X)$ *generate the closed cone of curves* $\overline{NE}(Fl(E))$ *of* $Fl(E)$.

Proof. To prove the claim, we will show that the curves $C_1, \ldots, C_\gamma, s(X)$ are dual to the generators $\tilde{\omega}_1, \ldots, \tilde{\omega}_\gamma, \mathcal{L}$ of the nef cone of $\mathrm{Fl}(E)$.

We have the following commutative diagram

$$\begin{array}{ccc} C_i & \overset{h}{\hookrightarrow} & \mathrm{Fl}(E) \\ {\scriptstyle (\Phi_i)_{res}}\downarrow & & \downarrow{\scriptstyle \Phi_i} \\ \Phi_i(C_i) & \overset{h'}{\hookrightarrow} & \mathrm{Gr}_{r_{k_i}}(E) \end{array}$$

Note that $(\Phi_i)_{res}$ is an isomorphism. Indeed, the following morphism is the inverse of $(\Phi_i)_{res}$ when restricted to $\Phi_i(C_i)$

$$\mathrm{Gr}_{r_{k_i}}(E_x) \to \prod_{i=1}^{\gamma} \mathrm{Gr}_{r_{k_i}}(E_x)$$

$$z \mapsto \left\{ [E_x \twoheadrightarrow W_{k_1}], \cdots, [E_x \twoheadrightarrow W_{k_{i-1}}], z, [E_x \twoheadrightarrow W_{k_{i+1}}], \cdots, [E_x \twoheadrightarrow W_{k_\gamma}] \right\}$$

Hence we have:

$$\begin{aligned} C_i \cdot \tilde{\omega_i} &= C_i \cdot \Phi_i^*(\omega_i) \\ &= \deg\left((\Phi_i \circ h)^*(\omega_i)\right) \\ &= \deg\left((h' \circ (\Phi_i)_{res})^*(\omega_i)\right) \\ &= \deg\left((h')^*(\omega_i)\right) \\ &= \Phi_i(C_i) \cdot \omega_i \\ &= 1. \end{aligned}$$

Similarly, using the following commutative diagram

$$\begin{array}{ccc} C_i & \xhookrightarrow{h} & Fl(E) \\ {\scriptstyle (\Phi_{i'})_{res}} \downarrow & & \downarrow {\scriptstyle \Phi_{i'}} \\ \{pt\} & \xhookrightarrow{h'} & Gr_{r_{k_i}}(E) \end{array}$$

we can see that $C_i \cdot \tilde{\omega_{i'}} = 0$ for $i \neq i'$.

Since the curve C_i is contained in a fiber $\pi^{-1}(x)$ for some $x \in X$, the composition map $C_i \hookrightarrow Fl(E) \to X$ is a constant map. Thus $C_i \cdot \mathcal{L} = 0$ for all i.

The sequence of quotients (3.2) defines the section $s(X)$ and the quotient $E \to E/E_{k_i}$ defines a section $s'(X)$ of $f_i : Gr_{r_{k_i}}(E) \to X$. Thus the image $\Phi_i(s(X))$ is precisely $s'(X)$. So we have the following commutative diagram:

$$\begin{array}{ccc} s(X) & \xhookrightarrow{s} & Fl(E) \\ {\scriptstyle (\Phi_i)_{res}} \downarrow & & \downarrow {\scriptstyle \Phi_i} \\ s'(X) & \xhookrightarrow{s'} & Gr_{r_{k_i}}(E) \end{array}$$

Then

$$s(X) \cdot \tilde{\omega_i} = \deg\left(s^*(\Phi_i^*(\omega_i))\right) = (s' \circ (\Phi_i)_{res})^*(\omega_i) = s'(X) \cdot \omega_i,$$

since $(\Phi_i)_{res} : s(X) \to s'(X)$ is an isomorphism.

So

$$s'(X) \cdot \omega_i = (\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) - \theta_i \mathcal{L}_i) \cdot s'(X) = \theta_i - \theta_i = 0.$$

The equality $\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) \cdot s'(X) = \theta_i$ can be seen as follows.

Let $p : Gr_{r_{k_i}}(E) \hookrightarrow \mathbb{P}(\wedge^{r_{k_i}} E)$ be the Plücker embedding. Let $s'' : X \to \mathbb{P}(\wedge^{r_{k_i}} E)$ be the section defined by the rank 1 quotient $E \to \wedge^{r_{k_i}}(E/E_{k_i})$. Then, by the projection formula, we obtain

$$\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) \cdot s'(X) = p^*\mathcal{O}_{\mathbb{P}(\wedge^{r_{k_i}} E)}(1) \cdot s'(X) = \mathcal{O}_{\mathbb{P}(\wedge^{r_{k_i}} E)}(1) \cdot p_* s'(X)$$
$$= \mathcal{O}_{\mathbb{P}(\wedge^{r_{k_i}} E)}(1) \cdot s''(X) = (s'')^*(\mathcal{O}_{\mathbb{P}(\wedge^{r_{k_i}} E)}(1)) = \deg(\wedge^{r_{k_i}}(E/E_{k_i}) = \theta_i.$$

Let $\pi_{\text{res}} : s(X) \hookrightarrow Fl(E) \to X$ be the composition map. Then π_{res} is an isomorphism with inverse given by the section map $s : X \to Fl(E)$. We then see

$$s(X) \cdot \mathcal{L} = \deg(\pi^*_{res}(\mathcal{L}')) = \deg(\mathcal{L}') = 1.$$

□

4. Seshadri constants of line bundles on Fl(E)

In this section, we prove our main results on Seshadri constants of nef line bundles on $Fl(E)$.

By Theorem 2.2, a nef line bundle L on $Fl(E)$ is of the form

$$a_1\tilde{\omega}_1 + a_2\tilde{\omega}_2 + \cdots + a_\gamma\tilde{\omega}_\gamma + b\mathcal{L},$$

for some non-negative integers $a_1, \ldots, a_\gamma, b$. We denote the line bundle L simply by the tuple $(a_1, a_2, \cdots, a_\gamma, b)$.

By Proposition 3.2, an effective curve C in $\overline{NE}(Fl(E))$ is of the form

$$p_1C_1 + p_2C_2 + \cdots + p_\gamma C_\gamma + rs(X),$$

for some non-negative integers $p_1, \ldots, p_\gamma, r$. We denote the curve C simply by the tuple $(p_1, p_2, \cdots, p_\gamma, r)$.

Lemma 4.1. *For each point y in $Fl(E)$ and for every $1 \leqslant i \leqslant \gamma$, there exist smooth curves $C'_i \subset Fl(E)$ passing through y which are numerically equivalent to the curve C_i.*

Proof. Let $\pi(y) = x$ in X. The lines C_i constructed at the beginning of Section 3 are in the fiber $Fl(E_x)$ over x.

The group $GL(E_x)$ acts transitively on $Fl(E_x)$. Let $C_i[0:1]$ be the point on the line C_i corresponding to $[0:1] \in \mathbb{P}^1$. Let g be in $GL(E_x)$ such that $g \cdot C_i[0:1] = y$.

Let C'_i be the image $g \cdot C_i$ of C_i under the linear automorphism g of $Fl(E_x)$. For any divisor D in $Fl(E)$, we will consider the divisor $D' = D \cdot Fl(E_x)$ in $Fl(E_x)$. Since C_i and C'_i are isomorphic as subschemes of $Fl(E_x)$ by a linear automorphism of $Fl(E_x)$, we have $C_i \cdot D' = C'_i \cdot D'$. □

Theorem 4.2. *Let X be a smooth complex projective curve and let E be a vector bundle on X which is not semistable. Let $Fl(E)$ be the flag bundle as in* (1.1).

Let $L = (a_1, a_2, \cdots, a_\gamma, b)$ be a nef line bundle on $Fl(E)$ expressed in terms of the generators of the nef cone given in Proposition 2.2. Then the Seshadri constants of L at any point $y \in Fl(E)$ satisfy the following inequalities:

$$\min(a_1, a_2, \cdots, a_\gamma, b) \leqslant \varepsilon(L, y) \leqslant \min(a_1, a_2, \cdots, a_\gamma).$$

Proof. Let $y \in \mathrm{Fl}(E)$. By Lemma 4.1, for every i there exist smooth curves C_i' passing through y which are numerically equivalent to the curves C_i. So for every $1 \leqslant i \leqslant \gamma$,

$$\frac{C_i' \cdot L}{\mathrm{mult}_x C_i'} = a_i. \tag{4.1}$$

This implies that $\varepsilon(L, y) \leqslant \min(a_1, a_2, \cdots, a_\gamma)$, giving one of the required inequalities.

Let $C \subset \mathrm{Fl}(E)$ be an irreducible and reduced curve passing through y. For the proof of the other inequality, we consider two cases.

Case 1: Suppose that C is not contained inside the fiber $\mathrm{Fl}(E_x)$ over the point $x := \pi(y) \in X$. Then by Bézout's theorem,

$$C \cdot \mathrm{Fl}(E_x) \geqslant \mathrm{mult}_y C \cdot \mathrm{mult}_y \mathrm{Fl}(E_x).$$

Since $\mathrm{Fl}(E_x)$ is a smooth variety, $\mathrm{mult}_y \mathrm{Fl}(E_x) = 1$. The associated line bundle to the divisor $\mathrm{Fl}(E_x) \subset \mathrm{Fl}(E)$ is $\mathcal{L}$. So

$$C \cdot \mathcal{L} \geqslant \mathrm{mult}_y C. \tag{4.2}$$

By Proposition 3.2, C is numerically equivalent to $p_1C_1 + p_2C_2 + \cdots + p_\gamma C_\gamma + rs(X)$ for some non-negative integers $p_1, \ldots, p_\gamma, r$. Then, by (4.2),

$$r \geqslant \mathrm{mult}_y C.$$

Then

$$\begin{aligned} \frac{C \cdot L}{\mathrm{mult}_y C} &= \frac{(p_1C_1 + p_2C_2 + \cdots + p_\gamma C_\gamma + rs(X)) \cdot (a_1\tilde{\omega}_1 + a_2\tilde{\omega}_2 + \cdots + a_\gamma\tilde{\omega}_\gamma + b\mathcal{L})}{\mathrm{mult}_y C} \\ &\geqslant \frac{p_1a_1 + \ldots + p_\gamma a_\gamma + rb}{r} \geqslant b. \end{aligned}$$

Case 2: Suppose now that the curve C is contained inside the fiber $\mathrm{Fl}(E_x)$ over x. Then C is numerically equivalent to $p_1C_1 + p_2C_2 + \cdots + p_\gamma C_\gamma$ for some non-negative integers $p_1, \ldots, p_\gamma$.

We have the following natural embedding of $\mathrm{Fl}(E_x)$. See (2.2).

$$\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_\gamma) : \mathrm{Fl}(E_x) \hookrightarrow \textstyle\prod_{i=1}^{\gamma} \mathrm{Gr}_{r_{k_i}}(E_x)$$

For $1 \leqslant i \leqslant \gamma$, the image $\Phi_i(C)$ of the curve C is contained in $\mathrm{Gr}_{r_{k_i}}(E_x)$. It is numerically equivalent to $p_i\Phi_i(C_i)$, where $\Phi_i(C_i)$ is a line in $\mathrm{Gr}_{r_{k_i}}(E_x)$.

Let $\mathcal{O}_{\mathrm{Gr}_{r_{k_i}}(E_x)}(1)$ be the tautological ample line bundle on the Grassmannian $\mathrm{Gr}_{r_{k_i}}(E_x)$. We will denote it by $\mathcal{O}_i(1)$, for simplicity. Then $\mathcal{O}_1(1) \boxtimes \mathcal{O}_2(1) \boxtimes \cdots \boxtimes \mathcal{O}_\gamma(1)$ is a very ample line bundle on $\prod_{i=1}^{\gamma} \mathrm{Gr}_{r_{k_i}}(E_x)$.

Let $\mathcal{O}_{\mathrm{Fl}(E_x)}(1) := \Phi^*(\mathcal{O}_1(1) \boxtimes \mathcal{O}_2(1) \boxtimes \cdots \boxtimes \mathcal{O}_\gamma(1))$, which is a very ample line bundle on $\mathrm{Fl}(E_x)$. Choose an effective Cartier divisor Y in the linear system $|\mathcal{O}_{\mathrm{Fl}(E_x)}(1)|$ such that

$y \in Y \cap C$ and $C \not\subseteq Y$. By Bézout's theorem,

$$C \cdot Y \geqslant (\mathrm{mult}_y C)(\mathrm{mult}_y Y).$$

Thus

$$\mathrm{mult}_y C \leqslant C \cdot Y. \tag{4.3}$$

For $1 \leqslant i \leqslant \gamma$, let H_i be a general hyperplane in $\mathrm{Gr}_{r_{k_i}}(E_x)$. Then we may write

$$Y = Y_1 + \ldots + Y_\gamma,$$

where

$$Y_1 \in |H_1 \times \mathrm{Gr}_{r_{k_2}}(E_x) \times \cdots \times \mathrm{Gr}_{r_{k_\gamma}}(E_x)|, \ldots,$$
$$Y_\gamma \in |\mathrm{Gr}_{r_{k_1}}(E_x) \times \mathrm{Gr}_{r_{k_2}}(E_x) \times \cdots \times \mathrm{Gr}_{r_{k_{\gamma-1}}}(E_x) \times H_\gamma|.$$

For $j \neq i \in \{1, \ldots, \gamma\}$, we have

$$C_j \cdot \left(\mathrm{Gr}_{r_{k_1}}(E_x) \times \mathrm{Gr}_{r_{k_2}}(E_x) \times \cdots \times H_i \times \cdots \times \mathrm{Gr}_{r_{k_\gamma}}(E_x)\right) = 0,$$

because the line C_j maps to a constant on $\mathrm{Gr}_{r_{k_i}}(E_x)$. Thus a general hyperplane H_i will not pass through this point.

On the other hand, for any $i \in \{1, \ldots, \gamma\}$,

$$C_i \cdot \left(\mathrm{Gr}_{r_{k_1}}(E_x) \times \mathrm{Gr}_{r_{k_2}}(E_x) \times \cdots \times H_i \times \cdots \times \mathrm{Gr}_{r_{k_\gamma}}(E_x)\right) = 1.$$

Hence

$$C \cdot Y = (p_1 C_1 + p_2 C_2 + \cdots + p_\gamma C_\gamma) \cdot Y = p_1 + p_2 + \cdots + p_\gamma.$$

Using (4.3), we obtain

$$\begin{aligned} \frac{C \cdot L}{\mathrm{mult}_y C} &\geqslant \frac{C \cdot L}{C \cdot Y} \\ &= \frac{a_1 p_1 + a_2 p_2 + \cdots + a_\gamma p_\gamma}{p_1 + p_2 + \cdots + p_\gamma} \\ &\geqslant \min(a_1, a_2, \cdots, a_\gamma). \end{aligned}$$

Combining the Cases 1 and 2, we conclude that every Seshadri ratio of L at y is at least b (happens in Case 1) or at least $\min(a_1, a_2, \cdots, a_\gamma)$ (happens in Case 2).

Thus $\varepsilon(L, x) \geqslant \min(a_1, a_2, \cdots, a_\gamma, b)$. This completes the proof of the theorem. □

Theorem 4.2 immediately gives the following.

Corollary 4.3. *Let the notation be as in Theorem 4.2. Let* $L = (a_1, a_2, \cdots, a_\gamma, b)$ *be a nef line bundle on* $Fl(E)$ *and suppose that* $b \geqslant \min(a_1, a_2, \cdots, a_\gamma)$. *Then* $\varepsilon(L, y) = \min(a_1, a_2, \cdots, a_\gamma)$ *for all* $y \in Fl(E)$.

We now show that the lower bound in Theorem 4.2 is achieved for some points in $\mathrm{Fl}(E)$.

Proposition 4.4. *At a point* y *in the section* $s(X)$, $\varepsilon(L, y)$ *will achieve the minimum value i.e.,* $\varepsilon(L, y) = \min(a_1, a_2, \cdots, a_\gamma, b)$.

Proof. Since we have $\varepsilon(L, y) \leqslant \min(a_1, a_2, \cdots, a_\gamma)$, it is enough to show that $\varepsilon(L, y) \leqslant b$ for any $y \in \mathrm{Fl}(E)$.

We see easily that the Seshadri ratio corresponding to the curve $s(X)$ is precisely b. Indeed, note that $s(X)$ is smooth since it is isomorphic to X. So

$$\frac{s(X) \cdot L}{\mathrm{mult}_y s(X)} = \frac{s(X) \cdot L}{1} = s(X) \cdot (a_1 \tilde{\omega}_1 + a_2 \tilde{\omega}_2 + \cdots + a_\gamma \tilde{\omega}_\gamma + b\mathcal{L}) = b.$$

This show that $\varepsilon(L, y) \leqslant b$ and proves the corollary. □

4.1. **Seshadri constants at general points of Fl**(E)**.** In this section, we show that the upper bound in Theorem 4.2 is achieved for a general point of $\mathrm{Fl}(E)$ under an additional assumption.

We quickly recall notation from Section 1. See (1.1).

Let

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E$$

be the Harder-Narasimhan filtration of E. Fix $1 \leqslant k_1 < k_2 < \cdots < k_\gamma \leqslant d-1$.

Then the flag bundle $\mathrm{Fl}(E) = \mathrm{Flag}(r_{k_1}, r_{k_2}, \cdots, r_{k_\gamma}, E)$ parametrizes flags of fibers of the form

$$E_x \to W_{k_1} \to W_{k_2} \to \cdots \to W_{k_\gamma},$$

where $\dim(W_{k_i}) = r_{k_i} := \mathrm{rank}\ E/E_{k_i}$, for $1 \leqslant i \leqslant \gamma$.

We now make the following additional assumption.

Assumption 4.5.

(1) For each $1 \leqslant i \leqslant \gamma$, there exists $c_i \in \{1, \ldots, d\}$ such that $\mathrm{rank}(E_{c_i}) = r_{k_i}$.
(2) For $1 \leqslant i \leqslant \gamma$, let $\zeta_i := \deg(E_{c_i})$. Then ζ_i is an integer multiple of r_{k_i}.

For the remainder of this section, we assume that Assumption 4.5 holds.

Next two results will be used for computing Seshadri constants at general points of $\mathrm{Fl}(E)$.

Proposition 4.6. [6, Theorem 4.1] *The pseudo-effective cone of* $Gr_{r_{k_i}}(E)$ *is generated by* $\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) - \zeta_i \mathcal{L}_i$ *and* $\mathcal{L}_i$.

Proposition 4.7. [5, Proposition 2.3] *The divisor* $\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) - \zeta_i \mathcal{L}_i$ *is effective and there exists a unique effective divisor in the linear system* $|\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) - \zeta_i \mathcal{L}_i|$, *i.e.,*

$$\dim H^0(\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) - \zeta_i \mathcal{L}_i) = 1.$$

The map Φ_i in the following commutative diagram is surjective.

$$\begin{array}{ccc} Fl(E) & \overset{\Phi}{\hookrightarrow} & \prod_{i=1}^{\gamma} Gr_{r_{k_i}}(E) \\ & \searrow^{\Phi_i} & \downarrow{\scriptstyle Pr_i} \\ & & Gr_{r_{k_i}}(E) \end{array}$$

Let D_i be the effective Cartier divisor on $Gr_{r_{k_i}}(E)$ corresponding to the line bundle $\mathcal{O}_{Gr_{r_{k_i}}(E)}(1) - \zeta_i \mathcal{L}_i$. Then the pullback $\Phi_i^{-1} D_i$ is an effective Cartier divisor in $|\Phi_i^*(Gr_{r_{k_i}}(E)(1) - \zeta_i \mathcal{L}_i)|$. By commutativity, we have $\Phi_i^{-1} D_i = \Phi^{-1}({Pr_i}^{-1}(D_i))$. The pullback of the divisor under the projection Pr_i is the effective divisor

$$Gr_{r_{k_1}}(E) \times Gr_{r_{k_2}}(E) \times \cdots \times D_i \times \cdots \times Gr_{r_{k_\gamma}}(E).$$

Thus the effective divisor $\Phi_i^{-1} D_i$ in $Fl(E)$ is

$$Fl(E) \cap \big(Gr_{r_{k_1}}(E) \times Gr_{r_{k_2}}(E) \times \cdots \times D_i \times \cdots \times Gr_{r_{k_\gamma}}(E)\big).$$

Remark 4.8. Note that $\dim H^0(Pr_i^*(Gr_{r_{k_i}}(E)(1) - \zeta_i \mathcal{L}_i)) = \dim H^0(Gr_{r_{k_i}}(E)(1) - \zeta_i \mathcal{L}_i) = 1$. We consider the restriction map of sections

$$H^0(Pr_i^*(Gr_{r_{k_i}}(E)(1) - \zeta_i \mathcal{L}_i)) \to H^0(Pr_i^*(Gr_{r_{k_i}}(E)(1) - \zeta_i \mathcal{L}_i)|_{Fl(E)})$$

This map is injective. If the only section s unique upto scaling maps to zero section, then $Fl(E)$ will be contained inside the divisor $Gr_{r_{k_1}}(E) \times Gr_{r_{k_2}}(E) \times \cdots \times D_i \times \cdots \times Gr_{r_{k_\gamma}}(E)$ which is not possible. The restriction of the section s is same as pulling back the section via the map Φ_i. So pullback of this section defines the above effective Cartier divisor in $Fl(E)$.

Define a closed subvariety $\mathcal{S} \subset Fl(E)$ as follows:

$$\begin{aligned} \mathcal{S} &= \cap_{i=1}^{\gamma}\big(Fl(E) \cap \big(Gr_{r_{k_1}}(E) \times Gr_{r_{k_2}}(E) \times \cdots \times D_i \times \cdots \times Gr_{r_{k_\gamma}}(E)\big)\big) \\ &= Fl(E) \cap \big(\cap_{i=1}^{\gamma} Gr_{r_{k_1}}(E) \times Gr_{r_{k_2}}(E) \times \cdots \times D_i \times \cdots \times Gr_{r_{k_\gamma}}(E)\big) \\ &= Fl(E) \cap D_1 \times D_2 \times \cdots \times D_\gamma. \end{aligned}$$

Note that $1 \leqslant \operatorname{codim}(\mathcal{S}, Fl(E)) \leqslant \gamma$ in $Fl(E)$.

Theorem 4.9. *Let the notation be as in Theorem 4.2. Assume that Assumption 4.5 holds. Let* $L = (a_1, a_2, \cdots, a_\gamma, b)$ *be a nef line bundle on Fl*(E). *If* $y \notin \mathcal{S}$, *then* $\varepsilon(L, y) = \min(a_1, a_2, \cdots, a_\gamma)$.

Proof. Since the point y does not belong to $\mathcal{S}$, let us assume without loss of generality that

$$y \notin D_1 \times Gr_{r_{k_2}}(E) \times Gr_{r_{k_3}}(E) \times \cdots \times Gr_{r_{k_\gamma}}(E).$$

Let C be a curve in $Fl(E)$ passing through y. Suppose that C is numerically equivalent to $p_1C_1 + p_2C_2 + \cdots + p_\gamma C_\gamma + rs(X)$, for some non-negative integers $p_1, \ldots, p_\gamma, r$.

If C is contained inside a fiber of $\pi : \mathrm{Fl}(E) \to X$, then by **Case 2** of Theorem 4.2,

$$\frac{C \cdot L}{\mathrm{mult}_y C} \geqslant \min(a_1, a_2, \cdots, a_\gamma).$$

We assume now that C is not contained inside a fiber. By Bézout's theorem we have $C \cdot \mathrm{Fl}(E_x) \geqslant \mathrm{mult}_y C$. Hence

$$C \cdot \mathrm{Fl}(E_x) = r \geqslant \mathrm{mult}_y C.$$

Since C contains y, C is not contained in $D_1 \times \mathrm{Gr}_{r_{k_2}}(E) \times \mathrm{Gr}_{r_{k_3}}(E) \times \cdots \times \mathrm{Gr}_{r_{k_\gamma}}(E)$.

Let $\Phi_1(C) \subset \mathrm{Gr}_{r_{k_1}}(E)$ be the projection of C under Φ_1. Then $\Phi_1(C)$ is numerically equivalent to $p_1\Phi_1(C_1)+rs(X)$, where $\Phi_1(C_1)$ is a line in $\mathrm{Gr}_{r_{k_1}}(E)$. The image of the section $s(X)$ under the map Φ_1 is the section in $\mathrm{Gr}_{r_{1_\gamma}}(E)$ defined by the quotient $E \to E/E_{k_1}$ and this image is isomorphic to $s(X)$. Note that

$$\theta_1 := \deg E/E_{k_1} = s(X) \cdot \mathcal{O}_{\mathrm{Gr}_{r_{k_1}}(E)}(1).$$

Since $\Phi_1(C)$ is not contained in the effective Cartier divisor D_1, $\Phi_1(C) \cdot D_1 \geqslant 0$. So

$$(p_1\Phi_1(C_1) + rs(X)) \cdot (\mathcal{O}_{\mathrm{Gr}_{r_{k_1}}(E)}(1) - \zeta_1\mathcal{L}_1) = p_1 + r\theta_1 - r\zeta_1 = p_1 + r(\theta_1 - \zeta_1) \geqslant 0.$$

Then

$$p_1 + r(\theta_1 - \zeta_1) \geqslant 0 \Rightarrow p_1 \geqslant r(\zeta_1 - \theta_1).$$

Since $\omega_1 = \mathcal{O}_{\mathrm{Gr}_{r_{k_1}}(E)}(1) - \theta_1\mathcal{L}_1$ is nef, ω_1 is pseudo-effective. By Proposition 4.6, we can then write

$$\mathcal{O}_{\mathrm{Gr}_{r_{k_1}}(E)}(1) - \theta_1\mathcal{L}_1 = (\mathcal{O}_{\mathrm{Gr}_{r_{k_1}}(E)}(1) - \zeta_1\mathcal{L}_1) + m\mathcal{L}_1 \text{ , for an integer } m \geqslant 0.$$

Hence $m = \zeta_1 - \theta_1 \geqslant 0$. If $\zeta_1 - \theta_1 = 0$, then the nef cone and the pseudo-effective cone of $\mathrm{Gr}_{r_{k_1}}(E)$ are the same which implies that the vector bundle E is semistable, by [6, Corollary 4.3]. This contradicts our assumption. Thus $\zeta_1 - \theta_1 \geqslant 1$. Thus $p_1 \geqslant r(\zeta_1 - \theta_1) \geqslant r$. Then

$$\begin{aligned}
\frac{C \cdot L}{\mathrm{mult}_y C} &= \frac{a_1p_1 + \cdots + a_\gamma p_\gamma + br}{\mathrm{mult}_y C} \\
&\geqslant \frac{a_1p_1 + \cdots + a_\gamma p_\gamma + br}{r} \\
&\geqslant \frac{a_1r + \cdots + a_\gamma p_\gamma + br}{r} \\
&\geqslant a_1 \\
&\geqslant \min(a_1, a_2, \cdots, a_\gamma).
\end{aligned}$$

The proof is now complete by Theorem 4.2. □

Main results of this section are summarized in the following.

Corollary 4.10. *Let the notation be as in Theorem 4.2. Let* $L = (a_1, a_2, \cdots, a_\gamma, b)$ *be a nef line bundle on* $Fl(E)$*. Then the following statements hold.*

(1) $\min(a_1, a_2, \cdots, a_\gamma, b) \leqslant \varepsilon(L, y) \leqslant \min(a_1, a_2, \cdots, a_\gamma)$ *for all* $y \in Fl(E)$.
(2) $\varepsilon(L) = \min(a_1, a_2, \cdots, a_\gamma, b)$.
(3) *Assume that Assumption 4.5 holds. Then* $\varepsilon(L, 1) = \min(a_1, a_2, \cdots, a_\gamma)$.

Question 4.11. Does Theorem 4.9 hold without Assumption 4.5?

4.2. **Examples.** In this concluding section, we give some examples where our results apply. In all the examples, X denotes a connected smooth complex projective curve.

Example 4.12. Let $E = L_1 \oplus L_2 \oplus \mathcal{O}_X{}^{\oplus 3}$ be a vector bundle of rank 5 over X, where $\deg(L_1) = 1$, and $\deg(L_2) = 2$. Note that $\mu(E) = \frac{3}{5}$ and E is not semistable.

The Harder-Narasimhan filtration of E is the following

$$0 \subseteq L_2 \subseteq L_2 \oplus L_1 \subseteq E.$$

We consider the flag bundle $Fl(4, 3, E)$. The Picard rank of $Fl(4, 3, E)$ is 3. Theorem 4.2 gives bounds on Seshadri constants for any nef line bundle L on $Fl(4, 3, E)$. Further, Proposition 4.4 also gives the value of $\varepsilon(L)$. However Assumption 4.5 does not hold in this case.

Example 4.13. Let $E = L_1 \oplus L_2 \oplus \mathcal{O}_X{}^{\oplus 3}$ be a rank 5 vector bundle on X, where $\deg(L_1) = 1$, and $\deg(L_2) = -1$. Then $\mu(E) = 0$ and E is not semistable.

Then the Harder-Narasimhan filtration of E is the following

$$0 \subseteq L_1 \subseteq L_1 \oplus \mathcal{O}_X{}^{\oplus 3} \subseteq E.$$

Consider the flag bundle $Fl(4, 1, E)$. The Picard rank of $Fl(4, 1, E)$ is 3.

As above both 4.2 and Proposition 4.4 apply, but not Theorem 4.9.

Example 4.14. Let $E = L_1 \oplus \mathcal{O}_X{}^{\oplus 3} \oplus L_2$ be a vector bundle of rank 5 over X, where $\deg(L_1) = 4$, and $\deg(L_2) = -1$. Here $\mu(E) = \frac{3}{5}$ and E is not semistable.

The Harder-Narasimhan filtration of E is the following

$$0 \subseteq L_1 \subseteq L_1 \oplus \mathcal{O}_X{}^{\oplus 3} \subseteq E.$$

We consider the flag bundle $Fl(4, 1, E)$. The Picard rank of $Fl(4, 1, E)$ is 3. In this examples, all our results apply: 4.2, Proposition 4.4 and Theorem 4.9.

Example 4.15. Let $E = L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus \mathcal{O}_X{}^{\oplus 3}$ be a vector bundle of rank 7 over X, where $\deg(L_1) = 3$, $\deg(L_2) = 1$, $\deg(L_3) = -1$, and $\deg(L_4) = -2$. Here $\mu(E) = \frac{1}{7}$ and E is not semistable.

The Harder-Narasimhan filtration of E is the following

$$0 \subseteq L_1 \subseteq L_1 \oplus L_2 \subseteq L_1 \oplus L_2 \oplus \mathcal{O}_X{}^{\oplus 3} \subseteq L_1 \oplus L_2 \oplus L_3 \oplus \mathcal{O}_X{}^{\oplus 3} \subseteq E.$$

In this case, we can consider several flag bundles which satisfy our set-up.

- All our results apply for the flag bundle $\mathrm{Fl}(2, 1, E)$.
- Some other flag bundles we may consider are given below. Assumption 4.5 does not apply to any of them.
  $\mathrm{Fl}(5, 1, E)$, $\mathrm{Fl}(5, 2, E)$, $\mathrm{Fl}(6, 5, E)$, $\mathrm{Fl}(6, 2, E)$, $\mathrm{Fl}(6, 1, E)$, $\mathrm{Fl}(5, 2, 1, E)$, $\mathrm{Fl}(6, 2, 1, E)$, or $\mathrm{Fl}(6, 5, 2, 1, E)$.

Example 4.16. Let $E = L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus {\mathcal{O}_X}^{\oplus 3}$ be a vector bundle of rank 7 over X, where $\deg(L_1) = 8$, $\deg(L_2) = 2$, $\deg(L_3) = -4$, and $\deg(L_4) = -5$. Here $\mu(E) = \frac{1}{7}$ and E is not semistable.

The Harder-Narasimhan filtration of E is the following

$$0 \subseteq L_1 \subseteq L_1 \oplus L_2 \subseteq L_1 \oplus L_2 \oplus {\mathcal{O}_X}^{\oplus 3} \subseteq L_1 \oplus L_2 \oplus L_3 \oplus {\mathcal{O}_X}^{\oplus 3} \subseteq E.$$

Assumption 4.5 holds for the flag bundle $\mathrm{Fl}(6, 5, 2, 1, E)$. So all our results are applicable in this example.

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