LEFSCHETZ PENCILS ON SURFACES AND AN APPLICATION TO SESHADRI CONSTANTS

KRISHNA HANUMANTHU AND A. J. PARAMESWARAN

Abstract. Let X be a nonsingular complex projective surface and let L be a very ample line bundle on X. We study the question: when does a general pencil in the linear system $|L|$ consist of only irreducible and reduced curves? We give a precise condition which guarantees an affirmative answer: in fact, this holds if and only if the genus of a general member of $|L|$ exceeds the irregularity of X. The proof is topological, using vanishing cycles and the theory of Lefschetz pencils. As an application, combined with a construction due to Miranda, we prove that for any positive rational number δ there exist a blow up $Y \to X$ at finitely many points, an ample line bundle M on Y and a point $x \in Y$ such that $\varepsilon(Y, M, x) = \delta$; Miranda proved this when $X = \mathbb{P}^2$.

1. Introduction

Let X be a smooth complex projective variety and let L be a nef line bundle on X. The *Seshadri constant* of L at $x \in X$ is defined as the real number

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\text{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves passing through x. Seshadri constants were defined by Demailly in [3], motivated by the Seshadri criterion for ampleness ([9, Theorem 7.1]) which says that L is ample if and only if $\varepsilon(X, L, x) > 0$ for all $x \in X$.

It is easy to see that Seshadri constants satisfy the upper bound $\varepsilon(X, L, x) \leqslant \sqrt[n]{L^n}$, where n is the dimension of X. As mentioned above, by the Seshadri criterion for ampleness, Seshadri constant of an ample line bundle at any point is positive. A lot of research is focused on finding better upper and lower bounds for Seshadri constants.

Ein-Lazarsfeld [5] gave non-trivial lower bounds at general points on surfaces. More precisely, they showed that if L is an ample line bundle on a surface X, $\varepsilon(X, L, x) \geqslant 1$ at a *very general* point x. On the other hand, Seshadri constants at arbitrary points can be arbitrarily small as shown by a construction of Miranda; see [5, Example 3.1]. Miranda produces such examples on suitable blow ups of the projective plane $\mathbb{P}^2$. Bauer [1, Proposition 3.3] extended this construction to surfaces of Picard rank 1.

Both these results use the same idea: start with a curve C which has a point of sufficiently high multiplicity, find a pencil of irreducible and reduced curves containing C and then blow up the surface at the base locus of the pencil. Then a suitably chosen line bundle will have the required small Seshadri constant. The key step is the construction of a pencil of irreducible and reduced curves. On $\mathbb{P}^2$ and more generally on surfaces of Picard number 1, this follows

Date: November 4, 2026.

2020 *Mathematics Subject Classification.* 14C20, 14D05, 14F45, 14N05, 32S30, 32S50.

Key words and phrases. Lefschetz pencils, Seshadri constants, Linear systems of curves, Vanishing cycles.

from a dimension count argument. For arbitrary surfaces, such dimension count arguments do not easily generalize.

In this note, we generalize the result to arbitrary surfaces. Since the dimension-count arguments used by Miranda and Bauer no longer apply, we instead study the topology of hyperplane sections directly, using vanishing cycles and the theory of Lefschetz pencils to construct the necessary pencil. We answer the following general question: if X is a smooth complex projective surface and L is a very ample line bundle on X, when does the linear system $|L|$ contain a pencil of irreducible and reduced curves. The answer is given in Theorem 3.1, which gives a topological criterion for irreducibility of singular members of Lefschetz pencils. More precisely, we show that the genus of a general member of $|L|$ is greater than the irregularity of X if and only if a general pencil in $|L|$ contains only irreducible and reduced curves. Using this result and Miranda's construction we prove our main result about Seshadri constants in Theorem 6.1.

Convex Fujita numbers of surfaces arise in the study of effective global generation of adjoint line bundles on surfaces. In [2], several results on these numbers are proved. One such result uses pencils of irreducible and reduced curves. We briefly discuss the relevant result of [2] and mention how we can establish further properties satisfied by the ample line bundles arising in this construction, in connection with convex Fujita numbers.

Section 2 proves a Bertini-type result which shows that any two hyperplane sections whose only singularity is a node are homeomorphic. Section 3 uses vanishing cycles to give a topological criterion for a general pencil to contain only irreducible and reduced curves. We also discuss when this topological condition fails. Section 4 constructs pencils with desired properties containing a given curve. Section 5 constructs curves with given multiplicity at a point, and Section 6 applies the topological constructions of the previous sections and Miranda's idea to obtain arbitrarily small Seshadri constants on blow ups of any surface.

2. Bertini-type results for singular hyperplane sections

Let $X \subset \mathbb{P}^n$ be a nondegenerate, smooth complex projective variety. We recall the definition of the dual variety and some of its properties below.

Let $\widetilde{\mathbb{P}^n}$ be the projective space of hyperplanes in $\mathbb{P}^n$. Consider the incidence variety given by

$$Y := \{(x, H) \in \mathbb{P}^n \times \widetilde{\mathbb{P}^n} \mid x \in X \cap H\}. \tag{2.1}$$

Then Y is nonsingular since it is a projective bundle over X. The fibres of the projection $p_2\big|_Y : Y \to \widetilde{\mathbb{P}^n}$ are the hyperplane sections of X in $\mathbb{P}^n$. So Y is also referred to as the universal hyperplane section over X.

We also define $Z \subset Y$ by

$$Z = \{(x, H) \in \mathbb{P}^n \times \widetilde{\mathbb{P}^n} \mid x \in X \cap H \text{ and } T_xX \subset T_xH \cong H\}.$$

Here T_xX denotes the tangent space of X at x. Note that Z is isomorphic to $\mathbb{P}(T_{\mathbb{P}^n}|_X/T_X)$, the projective normal bundle. Hence it is a nonsingular variety of dimension $n-1$.

One can check that Z is the critical locus of the map $p_2\big|_Y : Y \to \widetilde{\mathbb{P}^n}$; see [10, §2].

The *dual variety* $\check{X}$ of X is defined to be the image of Z in $\widetilde{\mathbb{P}^n}$ by the second projection. For $H \in \widetilde{\mathbb{P}^n}$, $X \cap H$ is nonsingular if and only if $H \notin \check{X}$.

When X is a surface we have the following result.

Theorem 2.1. [8, Corollary 7.4] *or* [4, Proposition 3.1]. *Let $X \subset \mathbb{P}^n$ be a nondegenerate, smooth complex projective surface. Assume that $n \geqslant 3$. Then the dimension of $\check{X}$ is $n-1$.*

Now let $X \subset \mathbb{P}^n$ be an arbitrary nondegenerate, smooth complex projective variety and assume that the dimension of $\check{X}$ is $n-1$.

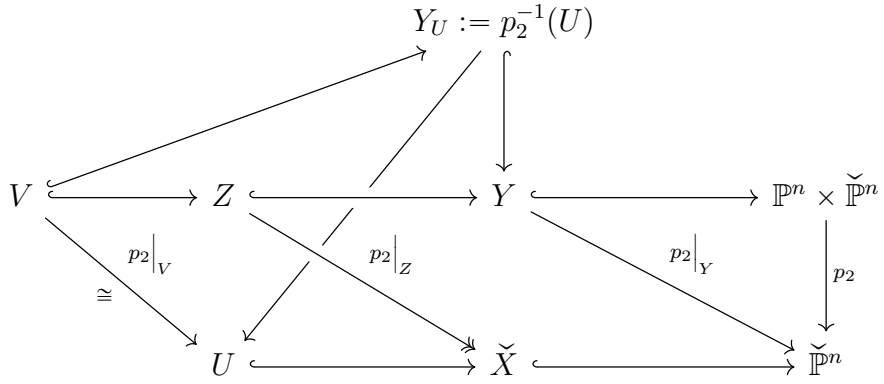
By Lefschetz double duality theorem [10, 2.2], the map $Z \to \check{X}$ is birational. Let

$$\begin{aligned} &V := \{(x, H) \in Z \mid \mathrm{Sing}(X \cap H) = \{x\} \text{ and } x \text{ is a nodal singularity of } X \cap H\}, \text{ and} \\ &U := p_2(V) \end{aligned} \tag{2.2}$$

Then V is a non-empty open subset of Z and U is a non-empty open subset of $\check{X}$. Further $p_2\big|_V : V \to U$ is an isomorphism and V is the maximal open subset of Z on which p_2 is an isomorphism. These statements follow from [11, Lemma 4.9].

Let $Y_U := p_2^{-1}(U)$.

The above set-up is summarized in the following diagram.



Note that the fibres of the map $Y_U := p_2^{-1}(U) \to U$ are singular along V and in fact they have nodal singularities along V. In an appropriate coordinate system on Y the singularity of a fibre of $Y_U \to U$ is locally given by $xy = 0$ and V is given by $x = y = 0$. So the pair (Y_U, V) is a Whitney stratification of the singular space Y_U.

Note that the map $Y_U \to U$ is proper. Further the restriction $Y_U \backslash V \to U$ is submersive because every fibre is smooth and the map $V \to U$ is an isomorphism. So the map $Y_U \to U$ is a proper stratified submersion with respect to the Whitney stratification. This implies that $Y_U \to U$ is a locally trivial fibration by Thom's First Isotopy Lemma [13, Proposition 11.1]. Since U is connected, all fibres of $Y_U \to U$ are homeomorphic.

As a consequence of this discussion, we obtain the following.

Theorem 2.2. *Let X be a nonsingular complex projective variety embedded non-degenerately in $\mathbb{P}^n$ and suppose that the dimension of $\check{X}$ is $n-1$. Let $X \cap H_1$ and $X \cap H_2$ be hyperplane sections each of which has a unique singularity which is a node. Then $X \cap H_1$ and $X \cap H_2$ are homeomorphic. In particular, general singular hyperplane sections are homeomorphic.*

3. Irreducibility of hyperplane sections in a general pencil

In this section, we prove our main result. It gives a criterion for the existence of a pencil of reduced and irreducible curves in the linear system of a very ample divisor on a nonsingular surface.

Theorem 3.1. *Let X be a nonsingular complex projective surface embedded non-degenerately in $\mathbb{P}^n$, with $n \geqslant 3$. Let $Y \subset \mathbb{P}^n \times \widetilde{\mathbb{P}^n}$ be as in* (2.1).

Then there exists a subset R of $\widetilde{\mathbb{P}^n}$ of codimension at least 2 such that the fibres of $p_2\big|_Y : Y \to \widetilde{\mathbb{P}^n}$ over points outside R are reduced and irreducible if and only if the following inequality holds for a general hyperplane H in $\mathbb{P}^n$:

$$dim\ H_1(X \cap H, \mathbb{C}) > dim\ H_1(X, \mathbb{C}). \tag{3.1}$$

Moreover, when the condition (3.1) *holds, R can be chosen such that the fibres over points outside R are either nonsingular or irreducible curves whose unique singularity is a node.*

Proof. By Theorem 2.1, $\dim(\check{X}) = n - 1$. Recall the non-empty open set $U \subset \check{X}$ defined in (2.2).

Let L be a general line in $\widetilde{\mathbb{P}^n}$. Then L intersects $\check{X}$ transversally and $L \cap \check{X} \subset U$. This can be seen by projecting from a general point outside $\check{X}$.

We claim that $R := \check{X} \backslash U$ satisfies the conclusions of the theorem.

Fix a point $H_0 \in U$ and let $X_0 := X \cap H_0$ be the corresponding hyperplane section so that X_0 has a unique singular point y_0, which is a node.

Since $R = \check{X} \backslash U$ is closed in $\check{\mathbb{P}}^n$ and $H_0 \notin R$, we may choose a ball $B \subset \check{\mathbb{P}}^n$ centered at H_0, of radius small enough that $B \cap \check{X} = B \cap U$ is given by $z_n = 0$ where $z_1, z_2, \cdots z_n$ are the coordinates of the ball B. Let $\mathcal{B} := p_2^{-1}(B) \subset Y$. Since $p_2\big|_Y$ is proper, the sets $p_2^{-1}(B')$, for balls $B' \subseteq B$ centered at H_0, form a fundamental system of neighbourhoods of X_0 in Y; shrinking B if necessary, we may therefore assume that X_0 is a strong deformation retract of $\mathcal{B}$; see [10, 5.3.1].

Choose $H_1 \in B \backslash \check{X}$ and put $X_1 := X \cap H_1$; this is a nonsingular curve, since H_1 does not lie on the dual variety. Following [10], we recall briefly how the node of X_0 determines a distinguished class of embedded circles on X_1.

Since $H_0 \in U$ and $B \cap \check{X} = B \cap U$, every fibre over B has at most one singular point, and that point is a node; moreover $p_2\big|_V : V \to U$ is an isomorphism, $B \cap \check{X}$ is a ball of dimension $n - 1$ and the family $\mathcal{B} \to B$ restricted to the curve $D := \{z_1 = z_2 = \cdots z_{n-1} = 0\}$ is a 1-parameter smoothing of X_0 and the total space of this smoothing is a smooth surface. Hence there are $r, \varepsilon > 0$ with $\varepsilon \ll r$, local analytic coordinates (x, y) on $p_2^{-1}(D)$ at $y_0 \in X_0$ defined on the ball $\{|x|^2 + |y|^2 < r^2\}$ centred at y_0, such that near y_0 the family is given by

$$x^2 + y^2 = t, \qquad |x|^2 + |y|^2 < r^2, \quad |t| < \varepsilon.$$

We may assume that $H_1 \in D$. We may also assume that the coordinate t on D vanishes at H_0, and the coordinate t_1 of H_1 is a positive real number. Thus X_0 is locally $\{x^2 + y^2 = 0\}$, the union of the two branches $x = \pm iy$ where i is a square root of -1.

The fibre $\{x^2 + y^2 = t_1\}$ with t_1 positive real can be described as follows in terms of real coordinates. Let $x = u_1 + iv_1$ and $y = u_2 + iv_2$ be the real representatives of the complex

variables. Then the fibre over a real number t_1 is expressed as

$$\{(u,v) \mid \ u_1^2 - v_1^2 + u_2^2 - v_2^2 = t_1 \quad u_1v_1 + u_2v_2 = 0 \quad u_1^2 + v_1^2 + u_2^2 + v_2^2 < r^2\}$$

When $t_1 < \varepsilon \ll r$ it follows that this fibre is isomorphic to a disc bundle as follows: consider the coordinates $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then the fiber is identified as

$$\{(u,v) \ \mid \ |u|^2 - |v|^2 = t_1 \leqslant |u|^2 + |v|^2 < r^2, \ \textit{and} \ u \cdot v = 0 \ \}$$

This is an annulus, identified as the disc bundle of the tangent bundle of the circle in the coordinates u and its tangent discs in the coordinates v. Hence it deformation retracts onto its core circle $|u|^2 = t_1$ given by

$$\delta_1 \ = \ \big\{u_1^2 + u_2^2 = t_1; \ \ v_1 = v_2 = 0\big\}.$$

This is an embedded circle in X_1 whose class generates the first homology of the annulus. We call δ_1 the *local vanishing cycle* at y_0: its radius $\sqrt{t_1}$ tends to 0 as $t_1 \to 0$, so δ_1 contracts to the node. In the total space, δ_1 bounds the *thimble*

$$\Delta_1 \ = \ \big\{u_1^2 + u_2^2 = t; \ \ v_1 = v_2 = 0; \ \ 0 \leqslant t \leqslant t_1\big\},$$

where $t = 0$ gives the single point y_0. Δ_1 is the cone over δ_1 with apex y_0, swept out by the vanishing cycles as t runs from t_1 to 0. Letting $t \to 0$ and using that X_0 is a deformation retract of $\mathcal{B}$, we conclude that X_0 is homotopy equivalent to $X_1 \cup_{\delta_1} \Delta_1$, i.e., X_0 is obtained from X_1 by attaching the 2-cell Δ_1 along the cycle δ_1; see [10, Figure 4] for a pictorial representation of this phenomenon. More details about this discussion can be found in [10, §5].

We now prove the following claim which relates the vanishing cycles to the irreducibility of $X \cap H_0$.

Claim: The class of the cycle δ_1 is nonzero in $H_1(X \cap H_1, \mathbb{C})$ if and only if $X \cap H_0$ is irreducible.

Proof of Claim: For simplicity, let $X_0 := X \cap H_0$ and $X_1 := X \cap H_1$. By the above discussion, X_0 is obtained from X_1 by attaching the 2-cell Δ_1 along the local vanishing cycle δ_1 and X_0 is homotopic to $X_1 \cup_{\delta_1} \Delta_1$.

By excision the homologies of the pair $(X_1 \cup \Delta_1, X_1)$ are isomorphic to the homologies of the pair (Δ_1, δ_1), which in turn are isomorphic to the homologies of the pair (D^2, S^1).

Hence $H_2(X_1 \cup \Delta_1, X_1) \cong H_2(\Delta_1, \delta_1)$ (by excision) is a $\mathbb{C}$-vector space of dimension one generated by (Δ_1, δ_1). Moreover, for $i \neq 2$, $H_i(X_1 \cup \Delta_1, X_1) = 0$.

Considering the long exact sequence associated to the pair $(X_1 \cup \Delta_1, X_1)$, we obtain

$$0 \to H_2(X_1, \mathbb{C}) \to H_2(X_0, \mathbb{C}) \to H_2(X_1 \cup \Delta_1, X_1) \cong \mathbb{C} \to H_1(X_1, \mathbb{C}) \to H_1(X_0, \mathbb{C}) \to 0.$$

Note that the boundary map $H_2(X_1 \cup \Delta_1, X_1) \cong \mathbb{C} \to H_1(X_1, \mathbb{C})$ sends the generator (Δ_1, δ_1) to δ_1.

Now if X_0 is reducible then $H_2(X_0, \mathbb{C})$ has dimension at least 2. On the other hand $H_2(X_0, \mathbb{C})$ fits in an exact sequence with initial term $H_2(X_1, \mathbb{C}) \cong \mathbb{C}$ and the next term $H_2(X_1 \cup \Delta_1, X_1) \cong \mathbb{C}$, hence has dimension at most 2. This implies that X_0 is reducible if and only if $H_2(X_0, \mathbb{C})$ has dimension 2, i.e., X_0 has exactly 2 irreducible components. Now $H_2(X_0, \mathbb{C}) \to \mathbb{C}$ is surjective as dimension of $H_2(X_0, \mathbb{C})$ is 2 and the kernel has dimension 1. Hence the class of δ_1 must be zero in $H_1(X_1, \mathbb{C})$ by the exactness of the sequence.

Conversely, if the class of δ_1 is zero in $H_1(X_1, \mathbb{C})$, then the map $\mathbb{C} \to H_1(X_1, \mathbb{C})$ is the zero map. Hence the dimension of $H_2(X_0, \mathbb{C})$ is 2 and X_0 is reducible. So the claim is proved.

Now recall that, by Theorem 2.2, the hyperplane sections corresponding to points in U are either all irreducible or all reducible, as they are homeomorphic to each other. So by the above claim, the classes of vanishing cycles in $H_1(X \cap H_1, \mathbb{C})$ are all zero or all non-zero.

Let C be a general fibre of p_2. We can transport the local vanishing cycles to C and this induces isomorphisms on H_1, so whether a homology class is zero does not depend on the choices made.

By [10, §7.2], the kernel of the map $H_1(C, \mathbb{C}) \to H_1(X, \mathbb{C})$ is generated by local vanishing cycles. By the Lefschetz hyperplane theorem, the map $H_1(C, \mathbb{C}) \to H_1(X, \mathbb{C})$ is surjective. Now we have the following:

$$\begin{aligned} \text{The fibres over } U \text{ are all reducible } &\overset{\text{Claim}}{\iff} \text{ classes of vanishing cycles in } H_1(C, \mathbb{C}) \\ &\qquad \text{are all zero} \\ &\overset{[10,\ §7.2]}{\iff} \text{ kernel of } H_1(C, \mathbb{C}) \to H_1(X, \mathbb{C}) \text{ is zero} \\ &\iff \dim H_1(C, \mathbb{C}) = \dim H_1(X, \mathbb{C}). \end{aligned}$$

Thus if $\dim H_1(C, \mathbb{C}) > \dim H_1(X, \mathbb{C})$, then all the fibres over U are irreducible. By the construction in Section 2, fibres over U have a unique singularity and it is a node. So $R = \check{X} \backslash U$ has the required properties.

Conversely, suppose that $\dim H_1(C, \mathbb{C}) = \dim H_1(X, \mathbb{C})$. Then the fibres over U are all reducible. Note that U is a dense open subset of $\check{X}$ and $\check{X}$ is a hypersurface in $\widetilde{\mathbb{P}^n}$ and if R is any subset of $\widetilde{\mathbb{P}^n}$ of codimension at least 2, then R cannot contain U. So the complement of R contains a point of U and the fibre over this point is reducible. Consequently R does not have the desired property.

This completes the proof. □

Remark 3.2. Theorem 3.1 has also been proved in [12, Proposition 3.2]. This paper in fact proves the statement in arbitrary characteristic. The proof in the complex case is similar to our proof.

Remark 3.3. A version of the statement of Theorem 3.1 has been proved in [2, Proof of Theorem 7.5]. In that paper it is shown that for a given very ample line bundle L on a smooth complex projective surface X, a suitable multiple of L contains a pencil of irreducible and reduced curves (for their argument, $3L$ works). In order to prove their result, they use the notion of 2-connected divisors. We note that our proof of Theorem 3.1 is very different as it uses topological methods. Moreover, we give an explicit criterion on L that ensures the existence of such pencils. In remarks below, we look at pairs (X, L) for which the required condition fails.

For more details on how the pencils of irreducible curves are used in [2], see Remark 6.5.

Example 3.4. Let $X = \mathbb{P}^2$ and $L = \mathcal{O}_X(2)$. Then a general pencil of conics in $\mathbb{P}^2$ contains a singular conic which must be union of two lines. Note that the inequality (3.1) fails for $(\mathbb{P}^2, \mathcal{O}_X(2))$. On the other hand, it is satisfied for $(\mathbb{P}^2, \mathcal{O}_X(d))$ when $d \geqslant 3$.

Example 3.5. It is interesting to study pairs (X, L) for which the inequality (3.1) fails. In other words, the following natural map is an isomorphism for a general curve $C \in |L|$:

$$H_1(C, \mathbb{C}) \to H_1(X, \mathbb{C}).$$

This is the case, for example, when $X = \mathbb{P}^2$ and $L = \mathcal{O}_X(1)$ or $L = \mathcal{O}_X(2)$. Another example is when X is a smooth quadric in $\mathbb{P}^3$ and L is the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$.

Also if B is a curve and V is a rank-2 vector bundle on B then $X = \mathbb{P}(V)$ and $L = \mathcal{O}_X(1) \otimes A$, where A is sufficiently ample line bundle on B will provide another example.

Remark 3.6. The inequality (3.1) holds for many pairs (X, L) of surfaces of X and very ample line bundles L on X. For instance, consider surfaces X of non-negative Kodaira dimension ($\kappa(X) \geqslant 0$) such that $q := h^1(X, \mathcal{O}_X) \leqslant 1$. Note that $\dim H_1(X, \mathbb{C}) = 2q$. Let L be a very ample line bundle on X and let $C \in |L|$ be a general curve. Let g be the genus of C. Then $\dim H_1(C, \mathbb{C}) = 2g$.

By the adjunction formula, we obtain

$$C^2 + K_X \cdot C = 2g - 2$$

Since C is very ample and mK_X is effective for $m \gg 0$, $C^2 + K_X \cdot C > 0$. Hence $g \geqslant 2 > q$. So the inequality (3.1) holds for any surface with $q \leqslant 1$ and $\kappa(X) \geqslant 0$.

Examples of surfaces covered by Remark 3.6 include K3 surfaces, Enriques surfaces, and hypersurfaces in $\mathbb{P}^3$ of degree at least 4.

Remark 3.7. We can extend the argument in Remark 3.6 to surfaces X with $q = 2$ and $\kappa(X) \geqslant 0$. Let L be a very ample line bundle on X. As before, we have

$$L^2 + K_X \cdot L = 2g - 2.$$

We claim that $g > q = 2$, as required.

We first show that $L^2 \geqslant 3$. Indeed, L^2 is the degree of the embedding of X in a projective space $\mathbb{P}^n$ determined by L. We have $L^2 \geqslant 2$ since this embedding is nondegenerate. In fact, it is well-known that the degree is at least $n - 2 + 1 = n - 1$. So if $L^2 = 2$, then $n = 3$. So X is a quadric in $\mathbb{P}^3$, which is not possible since we assumed that $q = 2$. Note that the case of a quadric in $\mathbb{P}^3$ has been covered by Example 3.5. Hence $L^2 \geqslant 3$, as claimed.

Now since mK_X is effective for $m \gg 0$, we have

$$2g - 2 = L^2 + K_X \cdot L \geqslant L^2 \geqslant 3.$$

Hence $g \geqslant 3$, as required.

Examples of surfaces covered by Remark 3.7 include abelian surfaces.

Remark 3.8. If L is ample, but not very ample it is possible that (3.1) fails outside the cases covered in Example 3.5. For example, let $X = J(C)$ be the Jacobian of a genus 2 curve C and let Θ be the theta divisor on X. Then Θ is ample, but not very ample. Note that $\Theta^2 = 2$ and $h^0(X, \Theta) = 1$. Moreover, C is isomorphic to the unique curve in the linear system $|\Theta|$. So in this case, we get $g = q = 2$.

Remark 3.9. In fact, all *quasi-polarized* surfaces (X, L) (i.e., L is a nef and big divisor on a smooth complex projective surface X) which satisfy the equality $g = q$ have been completely classified. See [15, Theorem] for a classification. Alternatively, for the case $\kappa(X) \leqslant 1$ see [7, Theorem 3.1] and for the case $\kappa(X) = 2$ see [6, Theorem 4.2].

In this classification the only cases where L is very ample are the ones listed in Example 3.5.

Remark 3.10. Suppose that X is a smooth complex projective variety of dimension $n \geqslant 3$ and let L be a very ample line bundle on X. Let $Y \in |L|$ be a member of the linear system of L. Then a simple dimension count shows that if Y is reducible then the singular locus of Y has dimension at least 1: if $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 have same dimension as Y, then the dimension of $Y_1 \cap Y_2$ is at least $n-1+n-1-n = n-2 \geqslant 1$. Clearly $Y_1 \cap Y_2$ is contained in the singular locus of Y.

On the other hand, it is a standard fact that members of a general pencil of a very ample line bundle on a smooth variety have at worst isolated singularities. So on a smooth variety of dimension at least 3, a general pencil of hyperplane sections consists of irreducible and reduced members.

4. Construction of a general pencil containing a given curve

We now show that we can construct a pencil of reduced and irreducible curves containing a given curve.

Theorem 4.1. *Let X be a nonsingular complex projective surface embedded non-degenerately in $\mathbb{P}^n$. Suppose that the following inequality holds for a general hyperplane H in $\mathbb{P}^n$:*

$$h^1(X \cap H, \mathcal{O}_{X\cap H}) > h^1(X, \mathcal{O}_X).$$

Let $C \subset X$ be any hyperplane section. Then there exists a pencil $\mathcal{P}$ of curves on X satisfying the following:

(1) $C \in \mathcal{P}$.

(2) Every curve in $\mathcal{P}$ different from C is irreducible and reduced.

Moreover, if C is reduced then the pencil $\mathcal{P}$ can be chosen to additionally satisfy the following condition:

(3) Any two distinct curves in $\mathcal{P}$ intersect transversally.

Proof. Let $p \in \widetilde{\mathbb{P}^n}$ be the point corresponding to C. Consider the subset $R \subset \widetilde{\mathbb{P}^n}$ constructed in Theorem 3.1. Note that R has codimension at least 2. We claim that a general line in $\widetilde{\mathbb{P}^n}$ through p meets R at most at p.

Consider the projection from p:

$$\pi_p : \widetilde{\mathbb{P}^n} \backslash \{p\} \to \mathbb{P}^{n-1},$$

where the target space $\mathbb{P}^{n-1}$ is the projective space parametrizing the lines in $\widetilde{\mathbb{P}^n}$ through p. Let $\overline{R} \subset \mathbb{P}^{n-1}$ be the closure of the image of $R\backslash\{p\}$. Then $\overline{R}$ has codimension at least 1. Hence the complement of $\overline{R}$ in $\mathbb{P}^{n-1}$ is a nonempty open set. Choose a point in the complement and the corresponding line L in $\widetilde{\mathbb{P}^n}$. Then L passes through p and does not intersect $R\backslash\{p\}$.

So this gives the required pencil satisfying the conditions *(1)* and *(2)*.

Suppose further that C is reduced. Then a general hyperplane section meets C transversally. So intersecting C transversally is an open condition. Further if C and C' meet transversally, then any two distinct curves in the pencil spanned by C and C' also meet transversally. Hence a general line through p in $\widetilde{\mathbb{P}^n}$ corresponds to a pencil in X satisfying the condition *(3)*. □

5. An irreducible curve with prescribed multiplicity at a point

The following is well-known, but we include a proof for completeness.

Lemma 5.1. *Let X be a nonsingular complex projective surface with a very ample line bundle $\mathcal{O}_X(1)$ and let $x \in X$ be a point. Let m be a positive integer. Then there exists an integer d_0 such that for all $d \geqslant d_0$, the linear system $|\mathcal{O}_X(d)|$ contains an irreducible and reduced curve C with $\mathrm{mult}_x C = m$ and which is nonsingular on $X\backslash\{x\}$.*

Proof. Let $\mathcal{M}$ denote the ideal sheaf of x. Since $\mathcal{O}_X(1)$ is ample, there exists d_0 such that the sheaf $\mathcal{M}^m \otimes \mathcal{O}_X(d)$ is globally generated for $d \geqslant d_0$.

Consider the blow up $\pi : \tilde{X} \to X$ of X at x, with the exceptional divisor E. Consider the line bundle $L_d := \pi^\star \mathcal{O}_X(d) - mE$ on $\tilde{X}$. Then there is a natural surjective map

$$\pi^\star \mathcal{M}^m \to \mathcal{O}_{\tilde{X}}(-mE).$$

Twisting by $\pi^\star \mathcal{O}_X(d)$, we obtain a surjective map

$$\pi^\star(\mathcal{M}^m \otimes \mathcal{O}_X(d)) \to L_d.$$

Since $\mathcal{M}^m \otimes \mathcal{O}_X(d)$ is globally generated, so is $\pi^\star(\mathcal{M}^m \otimes \mathcal{O}_X(d))$. Then L_d is also globally generated for $d \geqslant d_0$ since it is a quotient of a globally generated sheaf.

We also note that L_d is ample for $d > m$. This follows, for example, since the Seshadri constant $\varepsilon(X, \mathcal{O}_X(1), x) \geqslant 1$ as $\mathcal{O}_X(1)$ is very ample. By choosing $d_0 > m$ if necessary, we conclude that L_d is ample and globally generated for $d \geqslant d_0$.

Since L_d is globally generated, by Bertini, a general curve $\tilde{C} \in |L_d|$ is smooth. Since L_d is ample, every curve in the linear system $|L_d|$ is connected. So $\tilde{C}$ is smooth and connected and hence it is irreducible and reduced. Then $C = \pi(\tilde{C})$ has the required properties: C is irreducible, reduced, and smooth outside x. Further, $\mathrm{mult}_x C = \tilde{C} \cdot E = m$. □

6. Seshadri constants

We now prove our main theorem about Seshadri constants.

Theorem 6.1. *Let X be a nonsingular complex projective surface and let δ be a positive real number. Then there exists a blow up $\pi : Y \to X$ of X at finitely many points in X, an ample line bundle L on Y and a point $x \in Y$ such that $\varepsilon(Y, L, x) < \delta$.*

Proof. Choose an integer $m \geqslant 2$ such that $\frac{1}{m} < \delta$. Let $C \subset X$ be an irreducible and reduced curve containing a point x of multiplicity m; such a curve exists by Lemma 5.1. More precisely, C is a section of a very ample divisor $\mathcal{O}_X(d)$ vanishing at x with multiplicity m. By increasing d if necessary, we may also assume that the inequality (3.1) holds. Now choose a pencil $\mathcal{P}$ which contains C and satisfies the properties of Theorem 4.1.

Let $C' \in \mathcal{P}$, $C' \neq C$ and let $C \cap C' = \{p_1, \ldots, p_r\}$. Note that $r = C \cdot C'$ and each p_i is a simple point of C and C'.

Let $\pi : Y \to X$ be the blow up of X at $\{p_1, \ldots, p_r\}$. Let $\widetilde{C}$ denote the strict transform of C and let $E_1 = \pi^{-1}(p_1)$. Let $L = a\widetilde{C} + E_1$, $a \geqslant 2$. We claim that L is ample and $\varepsilon(Y, L, x) \leqslant \frac{1}{m} < \delta$.

The rest of the proof follows from Miranda's construction, but we include the details here for completeness.

We use Nakai's criterion to show the ampleness of L. First note that

$$L^2 = a^2\widetilde{C}^2 + 2a\widetilde{C} \cdot E_1 + E_1^2 = 0 + 2a - 1 > 0.$$

Now let $D \subset Y$ be any irreducible and reduced curve. We show that $L \cdot D > 0$.

Note that the strict transform $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ is base point free and determines a morphism $\phi : Y \to \mathbb{P}^1$.

If D is a fibre of ϕ then D is contained in a member of the pencil $\widetilde{\mathcal{P}}$ and by hypothesis all curves in $\widetilde{\mathcal{P}}$ are reduced and irreducible. So D itself is inside $\widetilde{\mathcal{P}}$. Then $L \cdot D = L \cdot \widetilde{C} = 1$.

Suppose now that D is not a fibre of ϕ. Then $\phi(D) = \mathbb{P}^1$. Let $p \in \mathbb{P}^1$. Then $F = \phi^{-1}(p)$ is a member of the pencil $\widetilde{\mathcal{P}}$. Since D is not a fibre, D and F are distinct and have non-empty intersection. So $D \cdot F = D \cdot \widetilde{C} > 0$. We also claim that $D \cdot E_1 \geqslant -1$. Indeed, if $D \cdot E_1 < 0$, then E_1 is a component of D. Since D is irreducible and reduced, $D = E_1$. Then $D \cdot E_1 = E_1^2 = -1$. This gives $L \cdot D = aD \cdot \widetilde{C} + D \cdot E_1 > 0$.

This concludes the proof that L is ample.

Now note that $x \notin C \cap C' = \{p_1, \ldots, p_r\}$. By abuse of notation, we denote the point in Y above x also by x. Then

$$\varepsilon(Y, L, x) \leqslant \frac{L \cdot \widetilde{C}}{m} = \frac{1}{m} < \delta,$$

as required. □

Remark 6.2. In the situation of Theorem 6.1, we note that one in fact has the equality $\varepsilon(Y, L, x) = \frac{1}{m}$, where m is the multiplicity of C at x. We prove this by showing that the Seshadri ratio computed by any curve passing through x, different from $\widetilde{C}$, is larger than $\frac{1}{m}$.

Let $\widetilde{C} \neq D \subset Y$ be any irreducible and reduced curve containing x. Then we have

$$\widetilde{C} \cdot D \geqslant m\text{mult}_x D.$$

Further, since $x \in D$, D is not equal to E_1. Then

$$L \cdot D = a\widetilde{C} \cdot D + D \cdot E_1 \geqslant am(\text{mult}_x D).$$

Hence

$$\frac{L \cdot D}{\text{mult}_x D} \geqslant am > \frac{1}{m}.$$

Since it is obvious that $\varepsilon(dL, x) = d\varepsilon(L, x)$ for any ample line bundle L and any positive integer d, using Remark 6.2 we obtain:

Corollary 6.3. *Let X be a nonsingular complex projective surface and let δ be a positive rational number. Then there exists a blow up $\pi : Y \to X$ of X at finitely many points in X, an ample line bundle L on Y and a point $x \in Y$ such that $\varepsilon(Y, L, x) = \delta$.*

Remark 6.4. It is easy to see that if L is ample and globally generated, then $\varepsilon(L, x) \geqslant 1$ for all points x. Let m be any positive integer and let X be any surface. The construction in the proof of Theorem 6.1 gives an ample line bundle L on a suitable blow up of X such that kL is not globally generated for $k < m$.

Remark 6.5. The notion of a *convex Fujita number* is studied in [2]. Let X be a smooth complex projective surface. The convex Fujita number $\text{Fu}(X)$ is defined to be the smallest non-negative integer m such that for all $s \geqslant m$ and any ample line bundles $L_1, \ldots, L_s$, the

adjoint bundle $K_X + L_1 + \ldots + L_s$ is globally generated. Using Reider's result [14], one can show that $0 \leqslant \mathrm{Fu}(X) \leqslant 3$ for any surface X.

In [2, Theorem 7.5], the authors show that for any surface X, there is a suitable blow up X' of X at finitely many points such that $\mathrm{Fu}(X') = 3$. Their proof critically uses the existence of a pencil of reduced and irreducible curves.

Let C, C' be reduced and irreducible curves on X such that the pencil spanned by them consists of reduced and irreducible curves meeting transversally. Let $C \cap C' = \{p_1, \ldots, p_r\}$. The authors consider the blow up $X' \to X$ at $\{p_2, \ldots, p_r\}$. Let M be the strict transform of C. Then the authors prove that

(1) M is ample,
(2) $K_{X'} + 2M$ is not globally generated.

As a consequence, $\mathrm{Fu}(X') = 3$.

By using our result, we can obtain additional properties of M. As in the proof of Theorem 6.1, given an integer $m \geqslant 2$, let C be an irreducible and reduced curve which admits a point x of multiplicity m. Then, again as before, choose a pencil of irreducible and reduced curves containing C. Now we blow up the base locus of the pencil after removing a point as in [2]. Then the line bundle M (defined as the strict transform of C as above) will have the following properties in addition to (1) and (2).

(3) $\varepsilon(X', M, x) = 1/m$.
(4) kM is not globally generated for $k < m$.

Note that $K_{X'} + 3M$ is globally generated by [14].

Acknowledgements

The first author is supported by a grant from Infosys Foundation and ANRF MATRICS grant (ANRF/ARGM/2025/002374/MTR). The second author is supported by KIIFB funding. A part of this work was done when the second author visited Chennai Mathematical Institute as Infosys Chair Professor. We are grateful to Ankit Rai for pointing out the work of Y. Fukuma [6, 7], to Alex Küronya for informing us about the paper [2], and to Serge Lvovski for informing us about the papers [12, 15]. We also thank Mike Roth for useful discussions.

References

[1] T. Bauer, Seshadri constants on algebraic surfaces, Math. Ann. **313** (1999), no. 3, 547–583.
[2] J. Chen, A. Küronya, Y. Mustopa and J. Stix, Convex Fujita numbers and the Kodaira-Enriques classification of surfaces, Manuscripta Math. **177** (2026), no. 2, Paper No. 17, 31 pp.
[3] J.-P. Demailly, Singular Hermitian metrics on positive line bundles, in *Complex algebraic varieties (Bayreuth, 1990)*, 87–104, Lecture Notes in Math., 1507, Springer, Berlin.
[4] L. Ein, Varieties with small dual varieties, I, Invent. Math. **86** (1986), no. 1, 63–74.
[5] L. Ein and R. K. Lazarsfeld, Seshadri constants on smooth surfaces, Astérisque No. **218** (1993), 177–186.
[6] Y. Fukuma, On polarized surfaces (X, L) with $h^0(L) > 0$, $\kappa(X) = 2$, and $g(L) = q(X)$, Trans. Amer. Math. Soc. **348** (1996), no. 10, 4185–4197.
[7] Y. Fukuma, A lower bound for the sectional genus of quasi-polarized surfaces, Geom. Dedicata **64** (1997), no. 2, 229–251.
[8] W. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Algebraic geometry (Chicago, Ill., 1980), pp. 26–92, Lecture Notes in Mathematics, **862**, Springer, Berlin, 1981.

[9] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin, 1970.
[10] K. Lamotke, The topology of complex projective varieties after S. Lefschetz, Topology **20** (1981), no. 1, 15–51.
[11] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note Ser., 77 Cambridge University Press, Cambridge, 1984.
[12] S. Lvovski, On surfaces with zero vanishing cycles, Manuscripta Math. **145** (2014), no. 3-4, 235–242.
[13] J. Mather, Notes on topological stability, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 4, 475–506.
[14] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math. (2) **127** (1988), no. 2, 309–316.
[15] F. L. Zak, Surfaces with zero Lefschetz cycles, Mat. Zametki **13** (1973), 869–880; English transl., Math. Notes **13** (1973), 520–525.

Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India
Email address: krishna@cmi.ac.in

Kerala School of Mathematics, Kunnamangalam PO, Kozhikode, Kerala, 673571, India
Email address: param@ksom.res.in