SYZYGIES OF SOME GIT QUOTIENTS

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Dedicated to Prof. R. Sridharan on the occasion of his 80th birthday

Abstract. Let X be flat scheme over $\mathbb{Z}$ such that its base change, X_p, to $\overline{\mathbb{F}}_p$ is Frobenius split for all primes p. Let G be a reductive group scheme over $\mathbb{Z}$ acting on X. In this paper, we prove a result on the N_p property for line bundles on GIT quotients of $X_{\mathbb{C}}$ for the action of $G_{\mathbb{C}}$. We apply our result to the special cases of (1) an action of a finite group on the projective space and (2) the action of a maximal torus on the flag variety of type A_n.

Keywords: Syzygy, GIT quotient, flag variety

1. Introduction

Syzygies of algebraic varieties have been studied classically since the time of Italian geometers. For instance, the question of projective normality and normal presentation of embeddings of projective varieties in a projective space was studied in depth. The subject has been revived and there is much renewed interest since Green [7, 8] developed a homological framework which encompasses the classical questions. It was noted that projective normality and normal presentation were really properties of a graded free resolution and N_p property was defined as a generalisation of this property.

We briefly review the notion of N_p property.

Let k be an algebraically closed field of characteristic 0. All our varieties are projective, smooth and defined over k.

Let $\mathcal{L}$ be a very ample line bundle on a projective variety X. Then $\mathcal{L}$ determines an embedding of X into the projective space $\mathbb{P}\big(H^0(X,\mathcal{L})\big)$. We denote by S the homogeneous coordinate ring of this projective space. Then the *section ring* $R(\mathcal{L})$ of $\mathcal{L}$ is defined as $\bigoplus_{n=0}^{\infty} H^0(X,\mathcal{L}^{\otimes n})$ and it is a finitely generated graded S-module. One looks at the minimal graded free resolution of $R(\mathcal{L})$ over S:

$$... \to E_i \to ... \to E_2 \to E_1 \to E_0 \to R(\mathcal{L}) \to 0$$

where $E_i = \bigoplus S(-a_{i,j})$ for all $i \geq 0$ and $a_{i,j}$ are some nonnegative integers.

We say that $\mathcal{L}$ has N_0 property if $E_0 = S$. This simply means that the embedding determined by $\mathcal{L}$ is projectively normal (or $\mathcal{L}$ is *normally generated*).

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$\mathcal{L}$ is said to have N_1 property if $E_0 = S$ and $a_{1,j} = 2$ for all j. In this case, we also say that $\mathcal{L}$ is *normally presented.* Geometrically, this means that the embedding is cut out by quadrics.

For $p \geq 2$, we say that $\mathcal{L}$ has N_p property if $E_0 = S$ and $a_{i,j} = i + 1$ for all $i = 1, \ldots, p$.

Given a very ample line bundle $\mathcal{L}$, it is an interesting question to ask whether it has N_p property for a given p.

There is extensive literature on this question. The following is a sample of results. For line bundles on curves see [9], on surfaces see [4, 5]. For abelian varieties, see [27]. In [2], a general result is proved for very ample line bundles on projective varieties. [6] studies N_p property for rational surfaces and Fano varieties (which are varieties with an effective anticanonical line bundle).

In this paper, we are interested in the N_p property for line bundles on GIT quotients. More specifically, we consider varieties defined over $\mathbb{Z}$ and consider the descent of an ample line bundle to a GIT quotient. We obtain a general result on N_p property of this descent (Corollary 4.3) by using a cohomological criterion for N_p property. We prove the required vanishing results using Frobenius splitting methods (Theorem 3.2).

In [18], the authors consider the quotients of a projective space X for the linear action of finite solvable groups and for finite groups acting by pseudo reflections. They prove that the descent of $\mathcal{O}_X(1)^{\otimes|G|}$ is projectively normal. In [13], these results were obtained for every finite group but with a larger power of the descent of $\mathcal{O}_X(1)^{\otimes|G|}$. In this paper, we consider any finite group acting linearly on X and prove a general result on N_p property for the descent of $\mathcal{O}_X(1)^{\otimes|G|}$.

A question of Fulton concerns the N_p property of line bundles on flag varieties (cf. [2, Problem 4.5]). The special case of flag varieties of type A_n is considered in [21] and a general result is obtained. In line with this, we consider the GIT quotient of a flag variety of type A_n for the action of a maximal torus and we obtain a result on N_p property as an application of our main result.

The organisation of the paper is as follows:

Section 2 consists of preliminaries. Cohomology of line bundles on the quotient variety is studied in Section 3. In section 4, we prove N_p property for GIT quotients of varieties which are defined over $\mathbb{Z}$. We apply these results to the special case of finite group quotients in Section 5 and to the special case of GIT quotient of a flag variety of type A_n for the action of a maximal torus in section 6.

2. Preliminaries

Given a vector bundle F on a projective variety X that is generated by its global sections, we have the canonical surjective map:

$$H^0(F) \otimes \mathcal{O}_X \to F. \tag{2.1}$$

Let M_F be the kernel of this map. We have then the natural exact sequence:

$$0 \to M_F \to H^0(F) \otimes \mathcal{O}_X \to F \to 0. \tag{2.2}$$

Our goal in this paper is to study N_p property of line bundles on GIT quotients of projective varieties with some special property.

Theorem 2.1. [2, Lemma 1.6] *Let L be a very ample line bundle on a projective variety X. Assume that $H^1(L^{\otimes k}) = 0$ for all $k \geq 1$. Then L satisfies N_p property if and only if $H^1(\wedge^m M_L \otimes L^{\otimes n}) = 0$ for all $1 \leq m \leq p+1$ and $n \geq 1$.*

Remark 2.2. In characteristic zero, it suffices to prove $H^1(M_L^{\otimes m} \otimes L^{\otimes n}) = 0$ to obtain N_p property as the wedge product $\wedge^m M_L$ is a direct summand of the tensor product $M_L^{\otimes m}$.

Remark 2.3. In [2], this theorem was proved only assuming that L is ample and base point free. We will apply this result with only these hypotheses (cf. [22, §1.3, Page 509]).

The following lemma is very useful in proving surjectivity of multiplication maps of sections of line bundles. See [3, Proof of Lemma 1.4].

Lemma 2.4. *Let E and $L_1, L_2, ..., L_r$ be coherent sheaves on a variety X. Consider the multiplication maps*

$\psi : H^0(E) \otimes H^0(L_1 \otimes ... \otimes L_r) \to H^0(E \otimes L_1 \otimes ... \otimes L_r),$

$\alpha_1 : H^0(E) \otimes H^0(L_1) \to H^0(E \otimes L_1),$

$\alpha_2 : H^0(E \otimes L_1) \otimes H^0(L_2) \to H^0(E \otimes L_1 \otimes L_2),$

$...,$

$\alpha_r : H^0(E \otimes L_1 \otimes ... \otimes L_{r-1}) \otimes H^0(L_r) \to H^0(E \otimes L_1 \otimes ... \otimes L_r).$

If α_1,...,α_r are surjective, then so is ψ.

Proof. We have the following commutative diagram where id denotes the identity morphism:

$$\begin{array}{ccc}
H^0(E) \otimes H^0(L_1) \otimes ... \otimes H^0(L_r) & \xrightarrow{\alpha_1 \otimes id} & H^0(E \otimes L_1) \otimes H^0(L_2) \otimes ... \otimes H^0(L_r) \\
\downarrow \phi & & \downarrow \alpha_2 \otimes id \\
H^0(E) \otimes H^0(L_1 \otimes ... \otimes L_r) & & H^0(E \otimes L_1 \otimes L_2) \otimes H^0(L_3) \otimes ... \otimes H^0(L_r) \\
 & & \downarrow \alpha_3 \otimes id \\
\downarrow \psi & & \cdots \\
 & & \downarrow \alpha_{r-1} \otimes id \\
H^0(E \otimes L_1 \otimes ... \otimes L_r) & \xleftarrow{\alpha_r} & H^0(E \otimes L_1 \otimes ... \otimes L_{r-1}) \otimes H^0(L_r)
\end{array}$$

Since $\alpha_1, \alpha_2, ..., \alpha_r$ are surjective and this diagram is commutative, a simple diagram chase shows that ψ is surjective. □

The following result, known as Castelnuovo - Mumford lemma, will be used often in this paper.

Lemma 2.5. [24, Theorem 2] *Let E be an ample and base-point free line bundle on a projective variety X and let F be a coherent sheaf on X. If $H^i(F \otimes E^{-i}) = 0$ for $i \geq 1$, then the multiplication map*

$$H^0(F \otimes E^{\otimes i}) \otimes H^0(E) \to H^0(F \otimes E^{\otimes i+1})$$

is surjective for all $i \geq 0$.

3. Cohomology of the quotient variety

Let X be a flat scheme over $\mathbb{Z}$. Let p be a prime number and let $\overline{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. Let X_p denote the $\overline{\mathbb{F}}_p$-valued points of X. Let $X_{\mathbb{C}}$ denote the $\mathbb{C}$-valued points of X. We assume that $X_{\mathbb{C}}$ is a projective variety over $\mathbb{C}$ and that X_p are projective varieties over $\overline{\mathbb{F}}_p$ for all primes.

We assume that there is a sheaf $\mathcal{N}$ on X such that the base change of $\mathcal{N}$ to $X_{\mathbb{C}}$, $\mathcal{N}_{\mathbb{C}}$, (respectively, $\mathcal{N}_p$ on X_p, for all primes) is an ample line bundle.

Finally assume that X_p is Frobenius split for all primes.

Let G be a reductive (not necessarily connected) algebraic group scheme over $\mathbb{Z}$ acting on X such that the action map $G_{\mathbb{C}} \times X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ is a morphism. Assume that every line bundle on $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-linearised and that $(X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}})$ is nonempty. We assume that the above hypotheses also hold for base change over $\overline{\mathbb{F}}_p$ for all but finitely many primes.

Let $Y_{\mathbb{C}}$ denote the GIT quotient $G_{\mathbb{C}}\backslash\backslash(X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}})$. Similarly let Y_p denote the GIT quotient of X_p with respect to the G_p-linearised line bundle $\mathcal{N}_p$. We further assume that $\mathcal{N}_{\mathbb{C}}$ (respectively, $\mathcal{N}_p$) descends to $Y_{\mathbb{C}}$ (respectively, Y_p for all primes). Let $\mathcal{L}_{\mathbb{C}}$ (respectively, $\mathcal{L}_p$) denote the descent of $\mathcal{N}_{\mathbb{C}}$ to $Y_{\mathbb{C}}$ (respectively, $\mathcal{N}_p$ to Y_p).

For the preliminaries and notion of Geometric Invariant Theory, we refer to [25] and [26]. For the notion of Frobenius splitting, see [23].

Lemma 3.1. *$\mathcal{L}_{\mathbb{C}}$ and $\mathcal{L}_p$ are ample line bundles on $Y_{\mathbb{C}}$ and Y_p respectively.*

Proof. Let $\phi : (X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}}) \to Y_{\mathbb{C}}$ be the natural categorical quotient map and let $\phi^\star : Pic(Y_{\mathbb{C}}) \to Pic((X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}}))$ be the pullback map.

Since $\mathcal{N}_{\mathbb{C}}$ is a $G_{\mathbb{C}}$-linearised line bundle on $X_{\mathbb{C}}$, by [25, Theorem 1.10, Page 38], there is an ample line bundle $\mathcal{M}$ on $Y_{\mathbb{C}}$ such that the $\phi^\star(\mathcal{M}) = \mathcal{N}_{\mathbb{C}}^{\otimes n}$ for some $n > 0$.

Since $\phi^\star(\mathcal{L}_{\mathbb{C}}) = \mathcal{N}_{\mathbb{C}}$, $\mathcal{M} \otimes \mathcal{L}_{\mathbb{C}}^{-n}$ is in the kernel of $\phi^\star$. Since every line bundle on $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-linearised, $Pic((X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}})) = Pic_{G_{\mathbb{C}}}((X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}}))$. By [19, Proposition 4.2, Page 83], $\phi^\star$ is injective. Hence $\mathcal{M} = \mathcal{L}_{\mathbb{C}}^{\otimes n}$ and $\mathcal{L}_{\mathbb{C}}$ is ample.

Proof is similar for $\mathcal{L}_p$. □

Theorem 3.2. *With the notation as above, the following statements hold.*

(1) *$H^i(Y_{\mathbb{C}}, \mathcal{L}^e_{\mathbb{C}}) = 0$ for every $e > 0$ and $i \geq 1$,*
(2) *Assume that G_p is linearly reductive for all but finitely many primes. Then $H^i(Y_{\mathbb{C}}, \mathcal{L}^e_{\mathbb{C}}) = 0$ for every $e < 0$ and $i < d$, where d denotes the dimension of Y.*

Proof. Since $\mathcal{N}_p$ is ample, by Serre's vanishing theorem, there is a positive integer r such that $H^i(X_p, \mathcal{N}_p^{\otimes p^r}) = 0$ for $i \geq 1$.

Now we will use the Frobenius splitting property of X_p to prove (1) (cf. [23]). Let F denote the Frobenius morphism corresponding to prime p.

Tensoring the map $\mathcal{O}_{X_p} \longrightarrow F_*\mathcal{O}_{X_p}$ by $\mathcal{N}_p$ and noting that $\mathcal{N}_p \otimes F_*\mathcal{O}_{X_p} \cong F_*F^*\mathcal{N}_p = F_*\mathcal{N}_p^{\otimes p}$ (projection formula) we see that the following map is injective:

$$H^i(X_p, \mathcal{N}_p) \longrightarrow H^i(X_p, F_*\mathcal{N}_p^{\otimes p}) = H^i(X_p, \mathcal{N}_p^{\otimes p}).$$

Iterating this process, we conclude that the map $H^i(X_p, \mathcal{N}_p) \longrightarrow H^i(X_p, \mathcal{N}_p^{\otimes p^r})$ is injective.

Thus $H^i(X_p, \mathcal{N}_p) = 0$ for $i \geq 1$.

Since X is flat over $\mathbb{Z}$, using semicontinuity theorem [10, Theorem III.12.8], we conclude that $H^i(X_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}) = 0$ for $i \geq 1$ (cf. [1, Proposition 1.6.2]).

Proof of $H^i(X_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}^{\otimes e}) = 0$ for $e \geq 2$ and for every $i \geq 1$ is similar.

Since $(X_{\mathbb{C}})^{ss}_{G_{\mathbb{C}}}(\mathcal{N}_{\mathbb{C}})$ is nonempty, we have $H^i(Y_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}}^{\otimes e}) = H^i(X_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}^{\otimes e})^{G_{\mathbb{C}}}$ for every $i > 0$ and $e \geq 0$, by [29, Theorem 3.2.a].

Hence, by the above arguments, $H^i(Y_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}}^{e}) = 0$ for every $i > 0$ and $e \geq 0$. This proves (1).

Since X_p is Frobenius split and G is linearly reductive over $\overline{\mathbb{F}}_p$, using Reynolds operator, we see that Y_p is also Frobenius split. For a proof, see [17, Theorem 3.7].

It is well known that there is a positive integer r such that $H^i(Y_p, \mathcal{L}_p^{-p^r}) = 0$ for $i \neq d$.

Now the proof of (2) is similar to the proof of (1) using the Frobenius splitting property of Y_p.

This completes the proof of theorem. □

4. N_p PROPERTY

Let X be a flat scheme over $\mathbb{Z}$. We assume that the hypotheses stated at the begininning of Section 3 hold. For simplicity of notation in this section we use letters X, Y and $\mathcal{L}$ to denote $X_{\mathbb{C}}, Y_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{C}}$ respectively.

In this section we prove a result on N_p property for $\mathcal{L}$ using Theorem 2.1. By Remark 2.3, we need the assumption that $\mathcal{L}$ is ample and base point free. By Lemma 3.1, $\mathcal{L}$ is ample.

We assume further that $\mathcal{L}$ is base point free. Let $d = dim(Y)$.

In [4, Theorem 1.3], the authors prove a strong general result on N_p property for an ample and base point free line bundle $\mathcal{L}$ on a projective variety X The bounds obtained are expressed in terms of the dimension of X and regularity of $\mathcal{L}$. Regularity of $\mathcal{L}$ is a measure of vanishing of higher cohomology of powers of $\mathcal{L}$. This result is generalized to multigraded regularity in [11, Theorem 1.1].

Our next theorem follows from the above mentioned results. However we provide a proof here for the sake a self-contained exposition.

Theorem 4.1. *Let $m, a \geq 1$ be a positive integers. Then we have $H^i(Y, M_{\mathcal{L}^{\otimes a}}^{\otimes m} \otimes \mathcal{L}^{b-i}) = 0$ for $i \geq 1$ and $b > m + d$.*

Proof. We proceed by induction on m.

Let $m = 1$. Let $a \geq 1$ and $b > d + 1$. We first show that $H^1(Y, M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^{b-1}) = 0$.

Consider the sequence 2.2 with $F = \mathcal{L}^{\otimes a}$:

$$0 \to M_{\mathcal{L}^{\otimes a}} \to H^0(\mathcal{L}^{\otimes a}) \otimes \mathcal{O}_Y \to \mathcal{L}^{\otimes a} \to 0. \tag{4.1}$$

Tensoring with $\mathcal{L}^{\otimes b-1}$ and taking cohomologies, we get

$$H^0(\mathcal{L}^{\otimes b-1}) \otimes H^0(\mathcal{L}^{\otimes a}) \xrightarrow{\alpha} H^0(\mathcal{L}^{\otimes b+a-1}) \to H^1(M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^{\otimes b-1}) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^1(\mathcal{L}^{\otimes b-1}).$$

Since $H^1(\mathcal{L}^{\otimes b-1}) = 0$ by Theorem 3.2(1), if the map α is surjective then it follows that $H^1(M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^{\otimes b-1}) = 0$.

To prove surjectivity of α, we will use Lemma 2.4 and first prove that the following map is surjective:

$$\alpha_1 : H^0(\mathcal{L}^{\otimes b-1}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes b}).$$

For this we use Lemma 2.5. The needed vanishings are $H^i(\mathcal{L}^{b-1-i}) = 0$, for $i = 1, \ldots, d$. These follow from Theorem 3.2(1) because $b > d + 1$.

Similarly, we obtain the surjectivity of the maps:

$$\alpha_2 : H^0(\mathcal{L}^{\otimes b}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes b+1}),$$
$$\alpha_3 : H^0(\mathcal{L}^{\otimes b+1}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes b+2}),$$

and so on. By Lemma 2.4, α is surjective.

For $i > 1$, we twist (4.1) by $\mathcal{L}^{\otimes b-i}$ and consider the long exact sequence in cohomology :

$$H^{i-1}(\mathcal{L}^{\otimes a+b-i}) \to H^i(M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^{\otimes b-i}) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^i(\mathcal{L}^{\otimes b-i})$$

We get the desired vanishing by Theorem 3.2(1).

Now let $m > 1$ and suppose that the theorem holds for $m - 1$. Let $a \geq 1$ and $b > m + d$ be given. First let $i = 1$.

Tensor the sequence 4.1 with $M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{b-1}$ and take cohomology:

$$H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes b-1}) \otimes H^0(\mathcal{L}^{\otimes a}) \xrightarrow{\alpha} H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes a+b-1}) \to H^1(M_{\mathcal{L}^{\otimes a}}^{\otimes m} \otimes \mathcal{L}^{\otimes b-1}) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^1(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes b-1}).$$

The last term is zero by induction hypothesis. Note that the hypothesis required for b holds. Hence it suffices to show that α is surjective.

In order to show that α is surjective we will use Lemma 2.4 and first consider the following

$$\alpha_1 : H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{b-1}) \otimes H^0(\mathcal{L}) \to H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes b}).$$

By Lemma 2.5, this map surjects if $H^j(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{b-1-j}) = 0$ for $j = 1, \ldots, d$. Since $b-1 > m-1+d$, the required vanishing is clear from induction hypothesis applied to $m-1$.

Now consider the map:

$$\alpha_2 : H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^b) \otimes H^0(\mathcal{L}) \to H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes b+1}).$$

Using Lemma 2.5, as we did for α_1, we conclude that α_2 is surjective too. Iterating this we obtain surjectivity of α_i for all i and hence α is also surjective.

Finally for $i > 1$, we have:

$$H^{i-1}(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes a+b-i}) \to H^i(M_{\mathcal{L}^{\otimes a}}^{\otimes m} \otimes \mathcal{L}^{\otimes b-i}) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^i(M_{\mathcal{L}^{\otimes a}}^{\otimes(m-1)} \otimes \mathcal{L}^{\otimes b-i})$$

The middle term is zero because the other two are zero by induction.

This completes the proof of the theorem. □

Corollary 4.2. *Let X be a flat scheme over $\mathbb{Z}$ and let G be a reductive group scheme over $\mathbb{Z}$ acting on X. Suppose that all the hypotheses stated at the beginning of Section 3 hold. Let $\mathcal{L}$ denote the descent to $Y_{\mathbb{C}}$ of the ample line bundle $\mathcal{N}_{\mathbb{C}}$ on $X_{\mathbb{C}}$.*

Then $\mathcal{L}^{\otimes a}$ has N_p property for $a > p + d$.

Proof. By Theorem 2.1, $\mathcal{L}^{\otimes a}$ has N_p property if $H^1(M_{\mathcal{L}^{\otimes a}}^{\otimes m} \otimes \mathcal{L}^{\otimes an}) = 0$ for all $1 \leq m \leq p+1$ and $n \geq 1$.

We apply Theorem 4.1 with $m = 1, \ldots, p+1$ and the required vanishing follows immediately. For instance, to obtain the result for $m = p+1$ and $n = 1$, the required inequality to apply Theorem 4.1 is $b > p + 1 + d$ where $b - 1 = a$. This is equivalent to $a > p + d$, which is precisely our hypothesis. □

We get a stronger result when we assume that the top cohomology of the structure sheaf $\mathcal{O}_Y$ vanishes.

Corollary 4.3. *Let X be a flat scheme over $\mathbb{Z}$ and let G be a reductive group scheme over $\mathbb{Z}$ acting on X. Suppose that all the hypotheses stated at the beginning of Section 3 hold. Let $\mathcal{L}$ denote the descent to $Y = Y_{\mathbb{C}}$ of the ample line bundle $\mathcal{N}_{\mathbb{C}}$ on $X_{\mathbb{C}}$. Suppose further that $H^d(Y, \mathcal{O}_Y) = 0$.*

Then $\mathcal{L}^{\otimes a}$ has N_p property for $a \geq p + d$.

Proof. We argue just as in Corollary 4.2. We need to use a stronger version of Theorem 4.1, with the hypothesis $b \geq m+d$ in place of $b > m+d$. The proof is very similar to the proof of Theorem 4.1. For instance, in the base case ($m = 1$), we required vanishing $H^i(Y, \mathcal{L}^{b-i}) = 0$, for $i = 1, \ldots, d$. For $i < d$, we apply Theorem 3.2(1). For $i = d$, we use the hypothesis on the structure sheaf $\mathcal{O}_Y$. □

5. GIT quotients for the action of a finite group on a projective space

Let G be a finite group of order n. Let $\rho : G \to GL(V)$ be a representation of G over $\mathbb{C}$. G operates on the projective space $X = \mathbb{P}(V)$ and every line bundle on X is G-linearised. Let d be the dimension of X.

Note that for every point $x \in X$, the isotropy subgroup G_x of x in G acts trivially on the fiber of x in $\mathcal{O}_X(n)$. Hence, by [19, Prop 4.2, Page 83], $\mathcal{O}_X(n)$ descends to the GIT quotient $Y = G\backslash\backslash X^{ss}_G(\mathcal{O}_X(n))$. Let $\mathcal{L}$ denote the descent of $\mathcal{O}_X(n)$ to Y.

Let $x \in X$. Since G is finite, there is a $s \in H^0(X, \mathcal{O}_X(1))$ such that $s(gx) \neq 0$ for all $g \in G$. Let $\sigma = \prod_{g\in G} g.s$. Then $\sigma \in H^0(X, \mathcal{O}_X(n))^G$ and $\sigma(x) \neq 0$. Hence $\mathcal{L}$ is base point free. Also note that $\mathcal{L}$ is ample by Lemma 3.1.

Note that the statement of Theorem 3.2(1) holds in this case. The higher cohomologies of nonnegative powers of $\mathcal{O}_X(n)$ are clearly zero and hence the higher cohomologies of nonnegative powers of $\mathcal{L}$ are zero too (cf. [29, Theorem 3.2.a]).

Hence, by the Corollary 4.3, we have the following.

Theorem 5.1. *$\mathcal{L}^{\otimes a}$ has N_p property for any $a \geq p + d$.*

For $p = 0$, we deduce the following corollary. This result is new compared to [18] since it works for every group. This result is also new compared to [13] since the bound on degree is small.

Corollary 5.2. *$\mathcal{L}^{\otimes d}$ has N_0 property.*

6. GIT quotients for the action of a maximal torus on the flag variety

For the preliminaries on semisimple algebraic groups, semisimple Lie algebras and root systems, we refer to [14, 15]. For the preliminaries on Chevalley groups we refer to [28].

Let G be a semisimple Chevalley group over $\mathbb{C}$ of rank n. Let T be a maximal torus of G, B a Borel subgroup of G containing T, which are defined over $\mathbb{Z}$. Let $N_G(T)$ denote the normaliser of T in G. Let $W = N_G(T)/T$ denote the Weyl group of G with respect to T.

We note that G, T, B, W are all defined over $\mathbb{Z}$. Hence the flag variety G/B of all Borel subgroups of G and Schubert varieties are also defined over $\mathbb{Z}$ [28, Page 21]. Note that the base change of any Schubert variety to $\overline{\mathbb{F}}_p$ is Frobenius split (cf [23, Theorem 2, Page 38] or [1, Theorem 2.2.5, Page 69]).

We denote by $\mathfrak{g}$ the Lie algebra of G. We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the Lie algebra of T. Let R denote the roots of G with respect to T. Let $R^+ \subset R$ be the set of positive roots with respect to B. Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset R^+$ denote the set of simple roots with respect to B. Let $\langle ., . \rangle$ denote the restriction of the Killing form to $\mathfrak{h}$. Let $\check{\alpha}_i$ denote the coroot corresponding to α_i. Let $\varpi_1, \varpi_2, \cdots, \varpi_n$ denote the fundamental weights corresponding to S.

Let s_i denote the simple reflection in W corresponding to the simple root α_i. For any subset J of $\{1, 2, \cdots, n\}$, we denote by W_J the subgroup of W generated by s_j, $j \in J$. We denote the complement of J in $\{1, 2, \cdots, n\}$ by J^c. For each $w \in W$, we choose an element n_w in $N_G(T)$ such that $n_wT = w$. We denote the parabolic subgroup of G containing B and $\{n_w : w \in W_{J^c}\}$ by P_J. In particular, we denote the maximal parabolic subgroup of G generated by B and $\{n_{s_j}; j \neq i\}$ by P_i.

Let $X(B)$ denote the group of characters of B and let $\chi \in X(B)$. Then, we have an action of B on $\mathbb{C}$, namely $b.k = \chi(b^{-1})k,\ b \in B,\ k \in \mathbb{C}$. Consider the equivalence relation $\sim$ on $G \times \mathbb{C}$ defined by $(gb, b.k) \sim (g, k), g \in G, b \in B, k \in \mathbb{C}$. The set of all equivalence classes is the total space of a line bundle over G/B. We denote this G- linearised line bundle associated to χ by $\mathcal{L}_\chi$.

Let $G = SL(n + 1, \mathbb{C})$. Let J be a subset of $\{1, 2, \cdots n\}$ and let P_J be the parabolic subgroup of G corresponding to J. Since G is simply connected, every line bundle on G/P_J is G-linearised (cf. [19, 3.3, Page 82]).

Let W^{J^c} be the minimal representatives of elements in W with respect to the subgroup W_{J^c}. For $w \in W^{J^c}$, let $X(w) = \overline{BwP_J/P_J} \subset G/P_J$ be the Schubert variety corresponding to w. Note that $X(w)$ is T-stable. Hence restriction of any line bundle on G/P_J to $X(w)$ is T-linearised.

Let χ be a dominant character of T which is in the root lattice such that $\langle \chi, \alpha_j \rangle > 0$ for every $j \in J$. Let $w \in W^{J^c}$ be such that $X(w)^{ss}_T(\mathcal{L}_\chi)$ is nonempty. By [20, Theorem 3.10.a, Page 758], $\mathcal{L}_\chi$ descends to the GIT quotient $T\backslash\backslash X(w)^{ss}_T(\mathcal{L}_\chi)$. Let $\mathcal{N}_\chi$ denote the descent.

Since χ is in the root lattice, by [12, Theorem 2.3], for every $x \in X(w)^{ss}_T(\mathcal{L}_\chi)$, there is a T-invariant section s of $\mathcal{L}_\chi$ such that $s(x) \neq 0$. Hence $\mathcal{N}_\chi$ is base point free. Also $\mathcal{N}_\chi$ is ample by Lemma 3.1.

Theorem 6.1. *Let $Y = T\backslash\backslash X(w)^{ss}_T(\mathcal{L}_\chi)$ be the GIT quotient of $X(w)$ for the T-linearised line bundle $\mathcal{L}_\chi$ on $X(w)$. Let d be the dimension of Y. Let $\mathcal{N}_\chi$ be the descent of $\mathcal{L}_\chi$ to Y. Then $\mathcal{N}_\chi^{\otimes a}$ has N_p property for $a \geq p + d$.*

Proof. This follows from the above discussion and Corollary 4.3. □

Now let $X = G/P_J$. We apply this theorem to the inverse of the canonical line bundle K_X of X.

Let R^+_J denote the set of all positive roots β satisfying $\beta \geq \alpha_j$ for some $j \in J$. Let χ_J be the sum of all elements in R^+_J. Then, by equality (6) in [16, Page 229], we have, $K_X^{-1} = \mathcal{L}_{\chi_J}$. Note that $\langle \chi_J, \alpha_j \rangle > 0$ for every $j \in J$. Hence, by [16, Remark 1, Page 232], K_X^{-1} is ample.

By using similar arguments as above, K_X^{-1} descends to the GIT quotient $T\backslash\backslash X^{ss}_T(K_X^{-1})$. Let $\mathcal{L}$ denote the descent. Again using similar arguments as above, we see that $\mathcal{L}$ is ample and base point free.

Let $d = dim(X) - dim(T)$. We have

Corollary 6.2. *$\mathcal{L}^{\otimes a}$ has N_p property for $a \geq p + d$.*

Remark 6.3. For simple algebraic groups G of types different from A_n canonical line bundle of the flag variety G/B does not, in general, descend to the GIT quotient. For instance, if G

is of type B_3, the coefficient of simple root α_1 in the expression of 2ρ is 5. By [20, Theorem 3.10.b, Page 758], we see that the canonical line bundle of G/B does not descend to the GIT quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}(2\rho))$.

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