During my PhD, I have been working at the Chennai Mathematical Institute with my supervisor Prof. Govind Krishnaswami on problems in nonlinear dynamics and chaos with a focus on the three-body [1, 2] and three rotor [3, 4, 5] problems. In addition to these problems, I also have some interest in non-Euclidean geometry, nonlinear time-series analysis and dynamics on networks.

1 Geometrical approach to the planar three-body problem

In [1] and the expository review [2], we investigated the *planar* dynamics of three masses subject to either Newtonian or inverse-square potentials by treating the trajectories as reparametrized geodesics of the Jacobi-Maupertuis (JM) metric on the configuration manifold. Isometries of the JM metric allow us to define reduced dynamics on quotients of the configuration space, which are simpler to study, but nevertheless, encode significant information on the full dynamics. We used Riemannian submersions to find the JM metrics on these quotients and showed that this geometric reformulation regularizes collisions in the case of the $1/r^2$, but not in the case of the $1/r$ potential. In contrast with other regularizations [6, 7], this does not involve an extrapolation of the dynamics past a collision nor a change in dependent variables. Moreover, extending work of Montgomery [8], we proved the negativity of the scalar curvature on the configuration manifold and its quotients for equal masses. Sectional curvatures were also found to be largely negative, indicating widespread geodesic instabilities. An interesting direction for further research is to relate the local geodesic instabilities we find to medium- and long-time behavior as well as chaos. While it is still a challenge to do this in the three-body problem, we were able to find a simpler system where we could establish such a connection. This is the problem of three coupled rotors.

2 Instabilities, chaos and ergodicity in the three rotor problem

In the three rotor problem, three equally massive particles move on a circle subject to attractive cosine inter-particle potentials [3, 4, 5]. The quantum version of this problem is relevant to modeling a chain of (three or more generally n) coupled Josephson junctions [9] where the rotor angles are the phases of the superconducting order parameters associated to the segments between junctions. The problem of two rotors reduces to that of a simple pendulum, while the $n \to \infty$ limit is described by the sine-Gordon field [10]. Between these two extremes, we find that the dynamics of three rotors is rich and displays novel signatures of chaos. Upon passing to center of mass variables, the *relative* dynamics on a two-torus was shown to possess only one conserved quantity: the relative energy E in units of the coupling strength.

Three families of periodic solutions: We found two families of periodic orbits at all energies: (a) *Pendula* where two of the rotors form a 'molecule' with zero separation at all times and (b) *Isosceles breathers* where one rotor is always midway between the other two. Interestingly, we also found *non-rotating choreographies* up to moderate energies, where all rotors are equally separated in time.

Four signatures of onset of widespread chaos: We found that the relative dynamics on the two-torus displays order-chaos-order behavior: it is integrable at $E = 0$ and $E = \infty$ but displays a fairly sharp transition from regular to chaotic behavior as E is increased beyond $E_c \approx 4$ and a more gradual return to regularity as $E \to \infty$. We discover several manifestations of this transition to stochasticity: (a) a dramatic rise of the fraction of the area of the Hill region of Poincaré surfaces occupied by chaotic sections, (b) the spontaneous breaking of discrete symmetries of Poincaré sections present at lower energies, (c) the change in the sign of the curvature of the Jacobi-Maupertuis metric, which goes from being strictly positive for $E < 4$ to possessing both signs when $E > 4$ and (d) the accumulation of stability transitions in pendula at the libration to rotation threshold at $E = 4$.

Ergodicity in a band of global chaos and recurrence time statistics: Remarkably, we also found a band of global chaos $5.33 \leq E \leq 5.6$, where generic Poincaré sections were seen to fill up the entire Poincaré surfaces, leading us to conjecture ergodic behavior. In fact, the Liouville measure ensemble-average distribution functions of relative angles and angular momenta were shown to agree with the corresponding time-average distributions, providing evidence for ergodicity in this band of energies. Moreover, trajectories emanating from a small volume were shown to become uniformly distributed over the energy hypersurface indicating that the dynamics is mixing. Furthermore, the distribution of recurrence times to finite size cells was found to follow an exponential law with the mean recurrence time satisfying a scaling law with an exponent (equal to $2/3$) as expected from global chaos and ergodicity.

Future directions: We hope to obtain a better understanding of the physical mechanism underlying the onset of chaos and accumulation of stability transitions. The application of the methods of Chirikov and Greene should provide some insight. It would be nice to prove the accumulation of stability transitions in pendula as in Ref. [11] and establish its asymptotic periodicity on a log scale. Moreover, the nature of bifurcations and local scaling properties at these transitions are also of interest. In another direction we would like to examine our conjecture that the three rotor problem is *not* integrable at any energy other than $E = 0$ and ∞. Finally, we hope to investigate quantum manifestations of classical chaos and ergodicity in this problem. This could have implications for the physics of chains of coupled Josephson junctions.

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