Mid Semester Exam – Introduction to Mathematical Finance

1pm - 4pm.

1. **Interest Rate.** An investor deposits a principal $P$ at an annual interest rate $r > 0$. With $n$ compounding periods per year, the amount after 1 year is $A_n = \left(1 + \frac{r}{n}\right)^n \times P$.

   (a) Bank A and Bank B both provide the sam interest rate. However, Bank A offers more compounding period per year compared to Bank B. Where will you deposit your money ? Prove it.

   (b) From this definition, derive the continuous compounding limit. What happens to the principal after 1 year ?

   (c) What if the instantaneous interest rate is dependent on time $r(t)$. Write the formula for principal after one year under (a) continuous compounding convention, (b) half-yearly compounding convention.

2. **Payoff Diagrams of Option Combinations.** Let $S_T$ denote the price of the underlying asset at the time of maturity $T$. Let $K$ denote the strike price of an option. The payoff functions for basic European options are defined as: (a) Long Call Payoff: $C(S_T, K) = \max(S_T - K, 0)$ and (b) Long Put Payoff: $P(S_T, K) = \max(K - S_T, 0)$. For any "short" (sold) position, the payoff is the negative of the "long" (bought) position. The total payoff of a portfolio is the algebraic sum of the individual payoffs of its components. Taking a long view reflects a long-term positive view about the asset: you expect the price to increase in time, and therefore you buy it. Taking a short position reflects a short-term positive view (long-term negative view) about the asset: you expect the price to decrease in time, and therefore you sell it.

   For each of the following portfolios, draw the graph of the payoff as a function of the asset price $S_T$ at maturity. Clearly label all axes, strike prices, and slopes.

   (a) **(2 Marks) The Straddle:** Buying one call option and one put option, both with the same strike price $K$.

   (b) **(3 Marks) Synthetic Long:** Buying one call option and selling one put option, both with the same strike price $K$. Compare this graph to the payoff of simply holding the underlying asset.

   (c) **(3 Marks) Bull Call Spread:** Buying one call option with strike price $K_1$ and selling another call option with strike price $K_2$, where $K_2 > K_1$.

   (d) **(4 Marks) Extension: The Butterfly Spread:** An investor believes the price of an asset will remain very stable. They construct a "Butterfly Spread" using three strike prices: $K_1, K_2$, and $K_3$, such that $K_2$ is the midpoint ($K_2 = \frac{K_1+K_3}{2}$). The portfolio consists of:

      - Buying one call option with strike $K_1$
      - Buying one call option with strike $K_3$
      - Selling **two** call options with strike $K_2$

      Draw the resulting payoff and identify the range of $S_T$ where the payoff is non-zero.

3. **Path-Dependent Options in a Binomial Model.** Consider a two-period binomial model where each period is 6 months ($\Delta t = 0.5$ years). Initial stock price $S_0 = 100$. Up-factor $u = 3/2$ and down-factor $d = 2/3$. The risk-free interest rate is 8% per annum, compounded continuously.

   An investor is interested in a "Fixed-Strike Lookback Call Option" with a strike price $K = 105$ and a maturity of $T = 1$ year (2 periods). The payoff of this option at $T = 2$ is defined as:

   $$\text{Payoff} = \max(\max(S_0, S_1, S_2) - K, 0)$$

   where $\max(S_0, S_1, S_2)$ is the maximum price the stock achieved at any of the nodes ($t = 0, 0.5, 1$) along the realized path.

(a) **(4 Marks) State Space Tree:** Draw the binomial tree for the stock price $S$. For each of the four possible paths $(uu, ud, du, dd)$, calculate the final payoff of the Lookback Call at $t = 1$ year.

(b) **(3 Marks) Risk-Neutral Measure:** Calculate the risk-neutral probabilities $p$ and $q = 1 - p$.

(c) **(5 Marks) Valuation:** Compute the no-arbitrage price of this option at $t = .5year$ on the $u$ path.

(d) **(5 Marks) Valuation:** Compute the no-arbitrage price of this option assuming a one-period binomial model (with period = 1 year) using the *Replication Portfolio* method. Note that, here, the payoff will become $\max(\max(S_0, S_1) - K, 0)$. Argue why the price increases or decreases compared to a two-period model.

4. **Delta Hedging** is a strategy used to reduce the risk of an option position $F(S)$ by offsetting its sensitivity to changes in the price of the underlying asset $Delta = F'(S) = dF/dS$. The trader buys $-\Delta$ units in the underlying asset so that the portfolio becomes $F(S) - \text{Delta} \times S$. The price of the portfolio is now approximately insensitive to small price movements. As the underlying price changes, the option's delta also changes, so the hedge must be rebalanced periodically.

   A US-based investment bank (Bank U) and a European-based investment bank (Bank E) have both just sold an identical vanilla FX EUR/USD call option with strike 1.15 and maturity 1 year - the right to buy 1 EUR in exchange for exactly 1.15 USD - let its price be denoted $C$. Let Exchange Rate $USD/EUR$ denote the number of US Dollars (USD) required to purchase one Euro (EUR). Let the current the spot rate $s$ of $USD/EUR$ be 1.10, it means 1 EUR = 1.10 USD. The FX Delta is defined as $\frac{\partial C}{\partial s}$. Suppose its value was determined to be $\frac{\partial C}{\partial s} = .05$.

   (a) **(6 Marks) The US Bank Perspective:** Bank U accounts for its profit, loss, and portfolio value entirely in USD. To Bank U, EUR is a foreign asset. So, they want the portfolio price to be roughly constant in USD. What specific trade must Bank U execute to delta-hedge this short option position? State the currency and the amount.

   (b) **(6 Marks) The EU Bank Perspective:** Bank E accounts for its profit, loss, and portfolio value entirely in EUR. So, they want the portfolio price to be roughly constant in EUR. What specific trade must Bank E execute to delta-hedge this short option position? State the currency and the amount.

5. **Variance Reduction via Importance Sampling.** Suppose we want to estimate the value of the integral $I = \int_0^1 x^3\,dx$. We know analytically that the exact answer is **0.25**. But, we want to compute it via Monte Carlo method and estimate error.

   (a) **(3 Marks)** Suppose we use standard Monte Carlo integration by drawing a single sample $X$ from a uniform distribution $U(0, 1)$. Write down the estimator $Y_{MC}$ for the integral $I$. Calculate the exact variance of this estimator, $\text{Var}(Y_{MC})$.

   (b) **(1 Marks)** What if we sample 100 points $x_i$ from $i = 1$ to 100. Write down the estimator and its variance. How does error change when we sample 100 points instead of 1?

   (c) **(6 Marks)** We want to reduce the variance by using Importance Sampling. The intuition is that, $x^3$ is very close to 0 for small values of $x$ and grows very quickly as $x$ approaches 1. Uniform sampling wastes a lot of time sampling small values of $x$ that barely contribute to the integral. To fix this, we introduce Importance Sampling where we sample higher values of x more frequently. We choose a proposal probability density function $q(x) = 2x$ for $x \in [0, 1]$. Write down the new Importance Sampling estimator $Y_{IS}$ for a single sample $X \sim q(x)$ and compute its variance.

   (d) **(6 Marks)** Choose another proposal probability density function $q(x) = 3x^2$ for $x \in [0, 1]$. Write down the new Importance Sampling estimator $Y_{IS}$ for a single sample $X \sim q(x)$ and compute its variance. Compare the error between $q(x) = 2x$ and $q(x) = 3x^2$. Do you have an intuitive explanation for what should be the proposal probability density function to minimize the variance?

6. **(10 Marks) Coin Toss.** Consider a sequence of independent tosses of a fair coin, where each toss results in Heads (H) or Tails (T) with a probability of 1/2. Let $E_A$ be the expected number of tosses required to observe a specific pattern $A$.

   (a) Calculate $E_{\text{HHH}}$, the expected number of tosses until the sequence HHH appears.

   (b) Calculate $E_{\text{HTH}}$, the expected number of tosses until the sequence HTH appears.

   (c) Two students, Alice and Bob, are observing the coin tosses. Alice wins if the pattern HTH appears first, and Bob wins if the pattern HHH appears first. Determine the probability that Alice wins the game.