End Sem Exam for Intro to Math Finance

Timing: 930 am - 1pm

April 26, 2026

85 Marks.

1. **5 Marks. Linear Combinations of Brownian Motion.** Let $W_1 = \{W_1(t)\}_{t\geq 0}$ and $W_2 = \{W_2(t)\}_{t\geq 0}$ be two independent canonical (standard) Brownian motions defined on the same probability space. Let $\rho \in [-1, 1]$ be a constant. Define a new stochastic process $X = \{X(t)\}_{t\geq 0}$ by:
$$X(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t)$$
Is $X(t)$ a Brownian motion?

2. **13 Marks. Correlation of Stochastic Integrals.** Let $W(t)$ be a standard Brownian motion. Consider two random variables defined at a fixed time $T > 0$:
$$X = W(T) \quad \text{and} \quad Y = \int_0^T t\, dW(t)$$
   (a) **(1 Marks)** What is the expectations $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.
   (b) **(4 Marks)** Compute the expectation $\mathbb{E}[Y]$ and the variance $\mathrm{Var}(Y)$.
   (c) **(8 Marks)** Compute the covariance $\mathrm{Cov}(X, Y)$.

3. **8 Marks. Maximum of a Brownian Motion and the Effect of Drift.** Let $W = \{W(t)\}_{t\geq 0}$ be a canonical (standard) Brownian motion. Define the maximum $M(T) = \max_{0\leq t\leq T} W(t)$.
   (a) **(6 Marks)** Compute the probability $\mathbb{P}(M(T) > 1)$.
   (b) **(2 Marks)** Suppose we introduce a constant drift $\mu \neq 0$, defining a new process
$$dX(t) = \mu dt + dW(t).$$
   Let $M_X(T) = \max_{0\leq t\leq T} X(t)$. Does the probability $\mathbb{P}(M_X(T) > 1)$ increase or decrease compared to the driftless case?

4. **10 Marks. Stochastic Differential Equations.** Find the SDE satisfied by the following processes, where W_t is the standard Brownian motion process:
   (a) $X(t) = W_t^3$
   (b) $Z(t) = e^{t/2}\cos(W_t)$

5. **6 Marks.** Compute the expectation of $W(t)e^{W(t)}$.

6. **6 Marks. Black-Scholes PDE and Boundary Conditions.** The price $V(S, t)$ of a derivative written on a non-dividend-paying underlying stock S is governed by the Black-Scholes Partial Differential Equation (PDE):
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$
where r is the constant risk-free interest rate and σ is the constant volatility of the stock. While the PDE governs the dynamics of any derivative on the asset, the specific contract is defined by its boundary and terminal conditions.

   Consider a European call option on this stock with a strike price K and maturity T. Let $C(S, t)$ denote the price of this call option.

(i) State the terminal condition $C(S, T)$ at maturity.

(ii) Determine the appropriate spatial boundary conditions for the option price as $S \to 0$ and as $S \to \infty$ for any time $t < T$. Provide a brief financial justification for each spatial boundary condition.

7. **12 Marks. The Geometry of Correlation.** Consider three distinct financial assets. Let their returns be represented by random variables X_1, X_2, and X_3 with zero mean and unit variance. Suppose the correlation between X_1 and X_2 is given by $\rho_{12} = \cos(\alpha)$, and the correlation between X_2 and X_3 is given by $\rho_{23} = \cos(\beta)$ for some fixed angles $\alpha, \beta \in [0, \pi]$. Let the correlation between X_1 and X_3 be denoted by ρ_{13}.

   (a) **(10 Marks)** A fundamental property in linear algebra is that any valid correlation matrix must be positive semi-definite. By using this, find a bound on ρ_{13}

   (b) **(2 Marks)** Briefly explain the geometric intuition behind this exact result by treating the random variables X_1, X_2, and X_3 as unit vectors in $\mathbb{R}^3$.

8. **25 Marks. Continuous-Time Limit of the Binomial Model.** Consider an n-period binomial model for a stock price over a fixed time horizon T. Let $\Delta t = T/n$ be the length of each time step. The initial stock price is S_0. In each time step, the stock price can either move up by a factor u or down by a factor d, given by the Cox-Ross-Rubinstein specification:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u}.$$

   (a) Let the probability of going up be constant at each step and given by $p = \frac{1}{2} + \lambda\sqrt{\Delta t}$ for some constant λ. Let X_i be a random variable that takes the value 1 if the stock goes up in period i, and 0 if it goes down. Let $Y = \sum_{i=1}^{n} X_i$ be the total number of upward movements over the n steps.

      i. **(5 Marks)** What is the distribution of Y? Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

      ii. **(5 Marks)** Express the terminal log-return $\ln(S_T/S_0)$ in terms of Y, n, σ, and Δt.

   (b) **(5 Marks)** Compute the expected value $\mathbb{E}[\ln(S_T/S_0)]$ and the variance $\mathrm{Var}(\ln(S_T/S_0))$ in the limit as $n \to \infty$ (which implies $\Delta t \to 0$).

   (c) **(5 Marks)** Use the Central Limit Theorem to show that the terminal stock price S_T converges to a Geometric Brownian Motion, and write down the resulting continuous probability distribution for $\ln(S_T)$.

   Note: This continuous limit serves as an intuitive illustration of Girsanov's Theorem. Observe that changing the underlying probability measure (by altering λ) changes the expected return (drift) of the resulting process, but the volatility σ remains strictly unchanged.

   (d) **(5 Marks)** Let the risk-free interest rate be r, compounded continuously, so that the risk-free return over one period is $e^{r\Delta t}$. Let $\tilde{p}$ be the risk-neutral probability of an up movement in a single step.

      i. Compute $\tilde{p}$ to first order using Taylor expansion.

      ii. What value of λ would correspond to this $\tilde{p}$?

      iii. What is $\lim_{\Delta t \to 0} \tilde{p}$?

   Note: Taking the limit of the binomial option pricing formula under these scaled risk-neutral parameters is one way to derive the Black-Scholes pricing formula for a European call option.