

# End Sem Exam for Intro to Math Finance

Timing: 930 am - 1pm

May 4, 2025

85 Marks.

1. **10 Marks. Valuation of a cashflow.** Consider a three-year treasury bond with annual coupons with a face value of \$100. This bond will pay (a) a fixed coupon interest of 10%, once per year for 3 years and (b) at the end of maturity of three years, returns the face value of \$100. Let the risk-free interest rate be  $r = 10\%/year$ , compounded annually.

(a) Compute the present value of this bond.

(b) Suppose interest rate goes up to  $12\%/year$ , what would be the new price of the bond?

(c) Suppose interest rate goes down to  $8\%/year$ , what would be the new price of the bond?

2. **5 Marks. Path-Dependent Payoff in a Binomial Model.** Consider a two-period binomial model for a stock with each time period = 1 year. The initial price is  $S_0 = 100$ , and in each period, it can either go up by a factor of 1.2 or go down by a factor of 0.8. For example, after one time period,

$$S_1 = \begin{cases} 1.2 \cdot S_0 = 120 & \text{(up)} \\ 0.8 \cdot S_0 = 80 & \text{(down)} \end{cases}$$

The risk-free rate is  $r = 10\%/year$ , compounded annually.

Consider a European derivative  $H$  with a path-dependent payoff:

$$H = \begin{cases} 10 & \text{if the stock goes up in the first period and down in the second (up-down path)} \\ 0 & \text{otherwise.} \end{cases}$$

Compute the arbitrage-free price of the derivative  $H$  at time 0.

3. **15 Marks. Incomplete Market in a Trinomial Model.** Consider a one-period trinomial model. A stock has initial price  $S_0 = 100$ , and in one time step equal to one year, it can take one of three possible values:

$$S_1 = \begin{cases} 120 & \text{(up)} \\ 100 & \text{(middle)} \\ 80 & \text{(down)} \end{cases}$$

The risk-free interest rate is  $r = 10\%/year$ , compounded annually.

(a) Find all possible risk-neutral probabilities of the stock going up/middle/down, i.e., probabilities under which the discounted stock price process is a martingale.

*Hint: You know how to do it for a binomial model, where there is exactly one risk-neutral probability for no-arbitrage pricing. However, here, you need to find the entire range of probabilities that will work.*

(b) Consider a European derivative with payoff given by

$$H = (S_1 - 100)^2.$$

Compute the range of arbitrage-free prices for this derivative.

4. **20 Marks. Eigenvalues of a Correlation Matrix.** Let  $\Sigma$  be the  $3 \times 3$  symmetric matrix given by

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}.$$

- (a) Compute the eigenvalues of  $\Sigma$  in terms of  $\rho$ .
- (b) For what values of  $\rho$  is  $\Sigma$  a valid correlation matrix?
- (c) Take the general case of an  $n \times n$  matrix with diagonal elements unity and all off-diagonal elements  $\rho$ . What are the eigenvalues? For what values of  $\rho$  is this a valid correlation matrix?
5. **5 Marks. Scaled Brownian Motion.** Let  $W(t)$  be a standard brownian motion process. Take a constant  $c > 0$ . Define  $X(t) = \frac{1}{\sqrt{c}}W(ct)$ . Show that  $X(t)$  is a standard Brownian Motion.
6. **15 Marks. Stochastic Differential Equations.** Find the SDE satisfied by the following processes, where  $W_t$  is the standard brownian motion process:

(a)  $X(t) = W_t^2$

(b)  $Y(t) = \exp(W_t)$

(c)  $Z(t) = \sin(W_t)$

7. **15 Marks. Pattern showdown in coin tosses.** In this game, we repeatedly toss a fair coin and keep track of the resulting sequence of Heads (H) and Tails (T). Each player selects a pattern of a given length. The first player whose chosen pattern appears in the coin sequence wins. Suppose three players, A, B and C choose the following patterns: – A: HHH, B: THH, C: HTH.

- (a) If players A and B are playing the game, who is more likely to win? Can you compute the probability of each player winning?
- (b) What if B and C are playing ?