Problem set 4: Classical Mechanics: 2^{nd} module

Refresher course on classical mechanics and electromagnetism

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1. Consider a uniform square plate of side L and total mass M.

   (a) Select a convenient right-handed orthonormal coordinate system and draw a diagram of the plate and the coordinate axes.

   (b) Where is the center of mass located? What are its coordinates in the chosen system of coordinates?

   (c) Find all the matrix elements of the inertia matrix in a suitable basis with origin at the center of mass. Exploit the symmetries of the mass distribution on the plate to simplify the calculation.

   (d) Find the principal moments of inertia of the plate and check whether they satisfy/saturate the triangle inequalities.

   (e) Find the corresponding principal axes of inertia, are they uniquely determined? Clarify.

   (f) Is the square plate a symmetric top? Why?

2. Now consider a uniform solid cube of side L and mass M.

   (a) Choose a convenient coordinate system with origin at the center of mass and draw a figure of the cube and coordinate axes.

   (b) Find the matrix elements of the inertia matrix in a suitable basis.

   (c) Find the principal moments of inertia of the cube and establish whether it is a spherical top.

3. Consider a rigid body with mass density $\rho(\mathbf{r})$ and total mass M. Suppose it has two mutually orthogonal planes of reflection symmetry, i.e., the mass distribution is unchanged upon reflection in either plane. Examples are a conical top, a uniform right circular cylinder and a uniform right elliptical cylinder.

   (a) Set up (in a figure) a Cartesian coordinate system that is adapted to the above two planes of symmetry, which may be taken as the xz and yz planes. Write two formulas for the reflection invariance of $\rho(x, y, z)$.

   (b) Show that the center of mass must lie on the intersection of the two planes of symmetry. On which axis does the CM $(\bar{X}, \bar{Y}, \bar{Z})$ lie?

   (c) Show that in the above Cartesian basis, the inertia tensor $I_{ij} = \int d^3\mathbf{r}\rho(r^2\delta_{ij} - r_i r_j)$ is diagonal, and write integral expressions for the three principal moments of inertia and specify the corresponding principal axes. Begin by writing out the inertia matrix of 9 components and consider the individual integrals.

   (d) Now suppose we restrict to a rigid body that has an axis of rotational symmetry, e.g. a right circular cylinder or conical top. The mass distribution is symmetric under rotation by any angle about the axis. How many pairs of orthogonal planes of reflection symmetry does such a rigid body have? Draw a picture.

   (e) Use rotation-invariance by $\pi/2$ ($x' = -y, y' = x$) to show that the integrals for two of the principal moments of inertia of a rigid body with an axis of rotational symmetry are equal.

4. Consider force free motion of a symmetric top with $I_1 = I_2$, as discussed in the lecture. Suppose the axis of the top makes an angle $\theta \neq 0$ with the fixed direction of $\mathbf{L}$.

   (a) Find the angle α between the angular velocity vector $\mathbf{\Omega}$ and angular momentum vector $\mathbf{L}$ (α is half the opening angle of the cone swept out by $\mathbf{\Omega}$). Express α in terms of θ, the principal moments of inertia and the magnitude of angular momentum L. How does α depend on time and L?

   (b) Suppose $I_1 \to I_3$ so that the symmetric top becomes a spherical top. Based on our study of the spherical top, what do you expect to happen to α? Is this expectation fulfilled by the above formula for α?

   (c) It can be shown that to take the limit of a rigid rotator (starting from a symmetric top), $\cos\theta$ must tend to zero faster than I_3. Using this, find the limiting value of α for a rigid rotator. Does it agree with the value obtained in our study of a rigid rotator?

5. Euler angles θ, ϕ, ψ are defined in Landau & Lifshitz (or see the lecture notes). Express the generalized velocities $\dot\theta, \dot\phi, \dot\psi$ in terms of the angular velocity components $\Omega_1, \Omega_2, \Omega_3$.

6. Consider force free rotational motion of a symmetric top ($I_1 = I_2 \neq I_3$) described in terms of Euler angles. Let the co-rotating axes x, y, z by chosen along principal axes of inertia. We follow the notation adopted in the lecture and notes. Write the Lagrangian and find the momenta conjugate to the Euler angles and identify which are conserved. From the figure, identify which among p_θ, p_ϕ, p_ψ corresponds to the Z component of angular momentum L_Z, and which corresponds to the z component L_z.

7. Consider force-free rotational motion of a rigid rotator with principal moments of inertia $I_1 = I_2, I_3 = 0$ and use Euler angles to parametrize the orientation of the body relative to a lab frame (see figure). Recall that the components of angular velocity relative to a co-moving frame are given in terms of Euler angles and their derivatives:

$$\Omega_1 = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi, \quad \Omega_2 = -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi, \quad \Omega_3 = \dot\psi + \dot\phi\cos\theta. \tag{1}$$

   (a) Find a simple expression for the rotator Lagrangian in terms of coordinates and velocities by choosing co-moving axes along principal axes. Axis of rotator is along $z = x_3$.

   (b) What is the configuration space of a rigid rotator? Which Euler angles provide coordinates on the configuration space, and what are their ranges? Identify any redundant coordinates. By redundant we mean neither the coordinate nor the corresponding generalized velocity appears in the Lagrangian.

   (c) For the non-redundant coordinates, obtain conjugate momenta and identify cyclic coordinates.

   (d) Find Lagrange's equations and express them as second order differential equations for the non-redundant Euler angles.

   (e) Show that $\theta(t) = \omega t$, $\phi(t) = \phi_o$, constant and $\psi(t) = 0$ for constant ω is a solution to the equations of motion.

   (f) Based on our study of the rigid rotator and the above example solution, say what the trajectories look like on the configuration space of a rigid rotator.

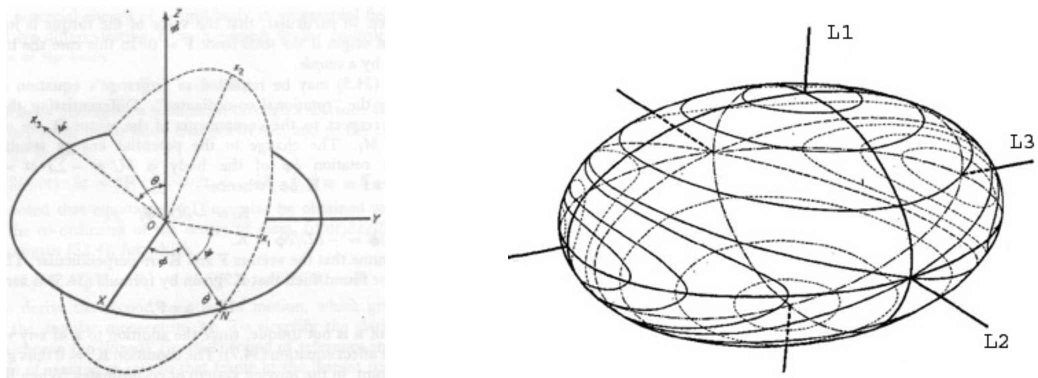

Figure 1: Euler angles and their time derivatives, from Landau and Lifshitz, Mechanics (fig. 47). Trajectories of Euler equations on an energy level surface for anisotropic top (from Landau & Lifshitz)

8. Consider free rotational motion of a symmetric top with principal moments of inertia $0 < I_1 = I_2 < I_3$. Denote the magnitude of angular momentum by $L = \sqrt{\mathbf{L} \cdot \mathbf{L}}$ and energy by E.

   (a) Write Euler's equations for the components of $\mathbf{L}$ with respect to a co-moving frame by choosing the co-rotating axes along principal axes of inertial. Use the common notation $a_{ij} = I_i^{-1} - I_j^{-1}$, $a = a_{23}$, $\omega = aL_3$.

   (b) Write down the general solution of Euler's equations.

   (c) Find all static (time-independent) solutions (L_1, L_2, L_3) of Euler's equations.

   (d) Draw a picture of the ellipsoid of inertia and mark five qualitatively distinct curves/points of intersection with angular momentum spheres.

   (e) Pick a static solution with minimal value of L for fixed E. Specify the solution (L_1, L_2, L_3) in terms of E and material constants. Comment on its stability with respect to a small increase in L.