Problem set 3: Classical Mechanics: 2^{nd} module
Refresher course on classical mechanics and electromagnetism
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1. Consider a particle of mass m moving subject to the double well potential $V(x) = g(x^2 - a^2)^2$ with $g, a > 0$.

   (a) Suppose we consider a non-static solution with energy $E = ga^4$, where the trajectory lies in the left well. Find the left turning point x_m of such a trajectory and indicate E, x_m in a graph of the potential.

   (b) Obtain the following expression for the time taken by the particle to go from x_m (starting at rest) to $x = 0$
   $$T = \sqrt{\frac{m}{2g}} \int_{x_m}^{0} \frac{dx}{\sqrt{2x^2a^2 - x^4}}. \tag{1}$$

   (c) Identify where in the interval $x_m \leq x \leq 0$ the integrand is singular (i.e. diverges). Roughly plot the integrand as a function of x in this interval.

   (d) Show that $T = \infty$ by considering the leading behaviour of the integrand near its singularities. Which singularity is integrable and which is not? Do this *without evaluating the indefinite integral explicitly*. Conclusion: a particle released from rest at x_m takes infinitely long to reach $x = 0$ and cannot cross the barrier.

2. Consider small transverse vibrations of a string stretched between a and b with constant mass per unit length ρ and constant tension τ, subject to Dirichlet boundary conditions. Suppose $u(x,t)$ and $\tilde{u}(x,t)$ are two solutions of the wave equation subject to the same initial conditions $u(x,0) = \tilde{u}(x,0) = h(x), \dot{u}(x,0) = \dot{\tilde{u}}(x,0) = v(x)$ and the same boundary conditions. Use conservation of energy to show that $u(x,t) = \tilde{u}(x,t)$, i.e., that the solution of the initial-boundary value problem for the wave equation is unique. Hint: Consider $w(x,t) = u(x,t) - \tilde{u}(x,t)$. What can you say about w?

3. Consider small transverse vibrations of height $u(x,t)$ of a string stretched between $x = a$ and $x = b$ with constant tension τ and mass per unit length ρ. Recall a typical Lagrangian from point particle mechanics $L_p = \frac{1}{2}m\dot{q}_i\dot{q}_i - V(\mathbf{q})$.

   (a) Provide a dictionary relating the following quantities from point particle mechanics to appropriate quantities for a vibrating string. (i) index i, (ii) coordinate q_i, (iii) particle mass m, (iv) $\sum_i$ (v) by analogy with the formula for momentum p_i, a formula for the momentum π conjugate to u.

   (b) Suppose $u(x,t) \to u(x,t) + \delta u(x,t)$ is an infinitesimal symmetry of the wave equation and boundary conditions. Then by analogy with the point particle case, propose a formula (without any proof) for the corresponding conserved quantity Q.

   (c) Show that a constant shift $u \to u + \alpha$ is a symmetry of the wave equation with open boundary conditions.

   (d) Using the previous proposal, give a formula for the corresponding Noether conserved quantity Q and check using the wave equation that it is indeed conserved.

4. It is possible to argue that the contraction of two ϵ symbols given below should be expressible as a linear combination of products of Kronecker deltas:

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = a\delta_{jk}\delta_{lm} + b\delta_{jl}\delta_{km} + c\delta_{jm}\delta_{kl} \quad \forall \quad 1 \leq j,k,l,m \leq 3. \tag{2}$$

   Find the constants a, b, c using the known properties and values of ϵ and δ.

5. We seek a generator of type $F_3(p, Q, t)$ for a finite canonical transformation from old to new canonical variables and hamiltonian $(q, p; H) \to (Q, P; K)$.

   (a) Staring from appropriate action principles for Hamilton's equations in the old and new variables, express the equations of transformation in terms of F_3, i.e., find q, P, K in terms of F_3

   (b) By comparing the relations among differentials for F_1 and F_3, express F_3 as a Legendre transform of $F_1(q, Q)$

   (c) Find a generating function of type $F_3(p, Q)$ that generates the scaling CT $Q = \lambda q.P = p/\lambda$.

6. Consider the finite canonical transformation, corresponding to a rotation of the phase plane

$$Q = cq - sp \quad \text{and} \quad P = sq + cp \quad \text{where} \quad s = \sin\theta \quad \text{and} \quad c = \cos\theta. \tag{3}$$

   (a) We seek a generating function of type-II $W(q, P)$ for the above finite CT. Find the differential equations that $W(q, P)$ must satisfy to ensure it generates the above CT.

   (b) Integrate the differential equations and give a simple formula for the generating function $W(q, P)$.

   (c) Verify that your proposed function $W(q, P)$ indeed generates the above finite rotation.

   (d) Find a generating function of type $F_1(q, Q)$ that generates the same finite rotation via an appropriate Legendre transform from $W(q, P)$. This provides an example of a CT that admits a generator of both type I and II.

   (e) Try to find a generator of type I for the identity CT, by letting the angle of rotation go to zero. What do you find?

7. Consider finite canonical transformations for one degree of freedom.

   (a) Find all (finite) *linear* canonical transformations $(q, p) \mapsto (Q, P)$ that fix the origin $(q = 0, p = 0)$. How many parameters are involved in their specification?

   (b) Identify the matrix group of such CTs. How many real parameters are involved in its specification?

   (c) Find a generating function of the second kind $W(q, P)$ for the above finite CTs. Express the answer in terms of the parameters used to specify the above CTs. Mention in what cases there is no generator of type 2 and give one such example with a suitable name.