Problem set 1: Classical Mechanics: 2^{nd} **module**

Refresher course on classical mechanics and electromagnetism
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1. Derive a conserved energy for Newton's equation for three degrees of freedom $m\ddot{x}_i = f_i$ where $i = 1, 2, 3$ or $m\ddot{\mathbf{r}} = \mathbf{f}$ where the cartesian components of the force are $f_i = -\frac{\partial V}{\partial x_i}$. Proceed by finding a suitable integrating factor.

2. Recall that for motion of a particle of mass m on a line, the solutions $x(t)$ of Newton's equation with energy E and initial position x_0 was reduced to the integral

$$t - t_0 = \pm \int_{x_0}^{x} \frac{dy}{\sqrt{\frac{2}{m}(E - V(y))}} \tag{1}$$

Consider a simple harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$ for which $E \geq 0$ and let $\omega = \sqrt{\frac{k}{m}}$.

   (a) Evaluate the integral (use $\int \frac{du}{\sqrt{1-u^2}} = \arcsin u$) and solve for the trajectories with given E, x_0. Show that you get

$$x(t) = \pm \sqrt{\frac{2E}{k}} \sin\left(\omega(t - t_0) \pm \arcsin\left(\sqrt{\frac{k}{2E}}\, x_0\right)\right). \tag{2}$$

   The upper signs correspond to one solution and the lower signs to another solution.

   (b) Specialize to the case where the particle starts from the equilibrium position at $t_0 = 0$ and simplify the formula for $x(t)$. Also find the momentum $p(t)$.

   (c) Show that $x(t)$ satisfies the 'initial conditions' $x(0) = 0$ and $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$.

   (d) For $E > 0$, indicate pictorially (in configuration space) how the two solutions obtained in the previous question differ.

   (e) On the x-p phase plane, draw a phase portrait for the simple harmonic oscillator indicating at least two qualitatively different trajectories and the arrow of time. Indicate the initial portions of the trajectories corresponding to the two solutions obtained above.

3. Practice with polar coordinates. Consider a particle moving on the x, y plane $z = 0$ in a central potential $V(r)$. The Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(r)$. Define plane polar coordinates for the particle's location via $x = r\cos\phi, y = r\sin\phi$. Abbreviate $\sin\phi = s, \cos\phi = c$. Recall that the unit vector in the radial direction is $\hat{r} = c\hat{x} + s\hat{y}$ and that linear momentum is $\mathbf{p} = m\dot{x}\hat{x} + m\dot{y}\hat{y}$. The Euler-Lagrange equations in polar coordinates were found to be $m\ddot{r} = mr\dot{\phi}^2 - V'(r)$ and $mr\ddot{\phi} = -2m\dot{r}\dot{\phi}$.

   (a) Show that $\dot{r} = c\dot{x} + s\dot{y}$

   (b) Show that $\dot{\phi} = \frac{1}{r^2}(x\dot{y} - y\dot{x})$.

   (c) Show that the momentum $p_r = m\dot{r}$ conjugate to r, is just the radial component of linear momentum $\mathbf{p} \cdot \hat{r}$.

(d) Show that the momentum $p_\phi = mr^2\dot\phi$ conjugate to ϕ is the z-component of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ by explicitly calculating the cross product.

(e) Draw the unit vector in the direction of increasing ϕ, called $\hat\phi$, in a diagram. Express $\hat\phi$ as a linear combination of $\hat x, \hat y$, using the diagram and an appropriate triangle. Choose $0 < \phi < \pi/2$. Check that $\hat r \cdot \hat\phi = 0$.

(f) Suppose we define the angular velocity as $\vec\omega = \dot\phi \hat z$ and a fictitious force $\vec F_c = -2\vec\omega \times \vec p$. Show that the $\hat\phi$ component of $\vec F_c$ is what appears on the rhs of the Euler-Lagrange equation $mr\ddot\phi = -2m\dot r\dot\phi$.

(g) $\hat x, \hat y$ are constant unit vectors, in the sense that they point in the same direction everywhere on configuration space and also at every point along a trajectory: $\frac{d\hat x}{dt} = \frac{d\hat y}{dt} = 0$. But $\hat r, \hat\phi$ change direction from place to place. Show that

$$\frac{d\hat\phi}{dt} = -\dot\phi\hat r \quad \text{and} \quad \frac{d\hat r}{dt} = \dot\phi\hat\phi \tag{3}$$

4. Consider a particle whose dynamics is specified by the Lagrangian $L(q, \dot q) = \frac{1}{2}m\dot q^2 + b(q)\dot q$. Here $b(q)$ is some differentiable function of q.

(a) Find the momentum conjugate to q and the equation of motion. What sort of motion does the Lagrangian describe?

(b) Explain the nature of this particle by examining the principle of extremal action $S = \int_{t_0}^{t_1} L\, dt$ for this Lagrangian. Can you relate this action to a more familiar one, how do they differ?

5. Recall that the general solution of $\ddot x = -\omega^2 x$ is $x(t) = a\cos\omega t + b\sin\omega t$ where a, b are constants of integration. Find the unique classical trajectory connecting $x(t_i) = x_i$ and $x(t_f) = x_f$ assuming $\omega\,\Delta t \neq n\pi$ for any integer n. Here $\Delta t = t_f - t_i$. You may use the abbreviations $c_i = \cos\omega t_i$, $s_f = \sin\omega t_f$ etc.

6. Consider a particle of mass m in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Suppose $x(t)$ is a trajectory between $x_i(t_i)$ and $x_f(t_f)$ and let $x(t) + \delta x(t)$ be a neighboring path with $\delta x(t_i) = \delta x(t_f) = 0$.

(a) Write the classical action of the path $x + \delta x$ as a quadratic Taylor polynomial in δx. Show that you get the following expression. What can you say about S_1?

$$S[x+\delta x] = S_0 + S_1 + S_2 = S[x] - \int_{t_i}^{t_f} (m\ddot x + m\omega^2 x)\,\delta x\, dt + \int_{t_i}^{t_f} \left[\frac{1}{2}m(\delta\dot x)^2 - \frac{1}{2}m\omega^2(\delta x)^2\right] dt$$

(b) For what values of κ is $x(t) + \delta x(t)$ a legitimate neighboring path for the variation

$$\delta x(t) = \epsilon\, \sin\kappa(t - t_i)\ ? \tag{4}$$

(c) Evaluate $S_2[\delta x]$ for all the allowed values of κ.

(d) Take $\Delta t = t_f - t_i = 10$s and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *less* than that of $x(t)$.

(e) Take $\Delta t = t_f - t_i = 10$s and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *more* than that of $x(t)$.

(f) What sort of an extremum of action is the classical trajectory?