Notes on Classical Mechanics: 2nd module

Phase space, Hamiltonian, Poisson brackets, Canonical transformations, Oscillations, Rigid bodies

Refresher course on classical mechanics and electromagnetism

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• These are lecture notes from the second week of the above Refresher Course. Please let me know (govind@cmi.ac.in) of any comments or corrections. Updated 27 Mar, 2020.

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1 Classical Mechanics: module 2 topics

Twelve 90 min meetings (lectures + tutorials): Phase space & portraits, Hamiltonian formulation, Small oscillations, normal modes, Poisson brackets & canonical transformations, Rigid body motion.

2 Books on Classical Mecahnics

1. Kittel, Knight, Ruderman, Helmholz and Moyer, Mechanics (Berkeley Physics Course Vol 1).
2. Kleppner and Kolenkov, An Introduction to Mechanics
3. A. P. French, Newtonian Mechanics.
4. Kibble and Berkshire, Classical Mechanics.
5. Hand and Finch, Analytical Mechanics.
6. Feynman, Leighton and Sands, Feynman lectures on physics.
7. K. R. Symon, Mechanics.
8. Goldstein, Poole and Safko, Classical Mechanics.
9. S. G. Rajeev, Advanced Mechanics.
10. Landau and Lifshitz, Vol 1, Mechanics

3 Review of Newtonian and Lagrangian mechanics of point particles

3.1 Configuration space, Newton's laws, phase space

• A point particle moving along a wire in the shape of a line or circle has one *degree of freedom*, namely its position (coordinate) along the wire. A point particle moving in a central force field has three degrees of freedom, we need three coordinates to specify the location of the particle. The Earth-moon system considered in isolation has six degrees of freedom. The number of degrees of freedom does not depend on the nature of forces. A rigid body like a duster has 6 degrees of freedom, three to locate its center of mass and three angles to orient it about the center of mass. A point-like molecule in the air has three degrees of freedom, e.g., its cartesian coordinates with respect to a chosen set of axes. N such molecules have $3N$ degrees of freedom. A fluid in a container has a very large number of degrees of freedom, say the locations of the

molecules, it is often modeled as a continuum, as a system with infinitely many degrees of freedom.

• An instantaneous configuration of the earth-moon system is any possible location of the earth and moon. The set of all instantaneous configurations of a mechanical system is called its configuration space $\mathcal{Q}$. For a particle on a plane, the configuration space is $\mathbb{R}^2$, two-dimensional Euclidean space. For a pair of point particles moving in space, $\mathcal{Q}$ is the space $\mathbb{R}^6$ with coordinates given (say) by the cartesian components of the radius vectors of each of the particles $\mathbf{r}_1^i, \mathbf{r}_2^j$ for $i, j = 1, 2, 3$. The number of degrees of freedom is the dimension of the configuration space.

• The zeroth law of classical mechanics can be regarded as saying that the trajectory $\mathbf{r}(t)$ of a particle is a (twice) differentiable function of time. This is a law that applies to planets, pendulums etc. But it fails for Brownian motion (movement of pollen grains in water). It also fails for electrons in an atom treated quantum mechanically. Newton formulated three laws of classical mechanics in the Principia.

• Newton's 1st law says that "Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed." [Isaac Newton, The Principia, A new translation by I.B. Cohen and A. Whitman, University of California press, Berkeley 1999.].

• The departure from rest or straight line motion is caused by forces. Newton's 2nd law says that the rate of change of momentum is equal to the impressed force, and is in the direction in which the force acts. For a single particle, the trajectory $\mathbf{r}(t) = (x^1, x^2, x^3) = (x, y, z)$ in *cartesian coordinates*, satisfies

$$m\ddot{\mathbf{r}} = \mathbf{F} \quad \text{or} \quad \dot{\mathbf{p}} = \mathbf{F}, \quad \text{or} \quad m\ddot{x}^i = F^i. \tag{1}$$

Here the momentum $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$. The trajectory $\mathbf{r}(t)$ is a curve on $\mathcal{Q}$ parameterized by time. Velocities $\dot{\mathbf{r}}(t)$ are tangent vectors to the trajectory. The form of Newton's equation changes in curvilinear coordinates, as we will see. Many interesting forces (such as gravity) arise as negative gradients of potential functions, $\mathbf{F} = -\nabla V(\mathbf{r})$. Force points in the direction of greatest decrease in potential energy

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} \tag{2}$$

E.g. $V = mgz$ for the gravitational potential energy and so $\mathbf{F} = -mg\hat{z}$ points downwards. For conservative forces, Newton's second law is

$$\dot{p} = -\nabla V \quad \text{or} \quad m\ddot{x}_i = -\frac{\partial V}{\partial x_i}. \tag{3}$$

For such 'conservative' forces, check that energy $E = \frac{1}{2}m\dot{\mathbf{r}}^2 + V(\mathbf{r})$ is conserved along trajectories $\dot{E} = 0$.

• One may wonder how this formula for energy arose from Newton's equation. Let us consider one degree of freedom. We wish to integrate $m\ddot{x} = -\frac{dV}{dx}$ with respect to time in order to solve the equation of motion. To do so we notice that $\dot{x}$ is an *integrating factor*. For, multiplying the equation by $\dot{x}$, both sides become total time derivatives:

$$m\ddot{x}\dot{x} = -\frac{dV}{dx}\frac{dx}{dt} \quad \text{or} \quad \frac{1}{2}m\frac{d\dot{x}^2}{dt} = -\frac{dV}{dt} \quad \text{or} \quad \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V\right) = \frac{dE}{dt} = 0. \tag{4}$$

So $E(t) = E(0)$ takes the same value as it initially had.

• Being 2nd order in time, Newton's equation requires both the initial position $\mathbf{r}$ and velocity/momentum $\dot{\mathbf{r}}$ or $\mathbf{p}$ as initial conditions. Knowledge of current position and momentum determines the trajectory via Newton's 2nd law. The *state* of the particle is specified by giving its instantaneous position *and* momentum. The set of possible instantaneous states of the particle is called its phase space M. For a particle moving on the real line, the phase space is $\mathbb{R}^2$ parametrized by the pair of coordinates (x, p). For a particle moving in 3D space, its configuration space is $\mathbb{R}^3$ and its phase space is $\mathbb{R}^6$ (locations and momenta).

• The path of the particle $\mathbf{r}(t)$ (satisfying Newton's equation and initial conditions) in configuration space is called its trajectory. Also of interest is the trajectory in phase space $(\vec{x}(t), \vec{p}(t))$. A phase portrait is a sketch of trajectories on phase space. Trajectories are oriented by arrows specifying forward time evolution.

• Consider the phase plane trajectories for a free particle with one degree of freedom. Since energy is conserved, phase space trajectories must lie inside level sets of energy $E = p^2/2m$. But in general, an energy level set[1] is a union of trajectories. For the free particle, the energy-E_0 level set is (in general) a pair of horizontal straight lines of fixed $p = \pm\sqrt{2mE}$ and arbitrary x. Trajectories come with a direction, the arrow of time. Draw the phase portrait.

• Newton's 3rd law says that to every 'action' there is always opposed an equal reaction. The sun attracts the Earth with a force equal in magnitude and opposite in direction to the force exerted by the Earth on the sun.

3.2 Energy, Angular momentum, conserved quantities, dynamical variables

• The energy of a particle moving in 3d space (say under the gravitational force) is a sum of kinetic and potential energies

$$E = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x, y, z) \tag{5}$$

$\dot{x} = \frac{dx}{dt}$ is the x-component of velocity. We often think of energy as a function of coordinates and momenta $E(x, y, z, p_x, p_y, p_z)$. So energy is a function on the phase space.

• Newton's equation implies that the energy is a constant of motion if the forces are conservative (expressible as the gradient of a potential). We say the energy is conserved in time/is a conserved quantity.

$$\dot{E} = m\sum_i \dot{x}_i\ddot{x}_i + \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = 0. \tag{6}$$

• Conserved quantities are useful. They help us solve/understand Newton's equation for the trajectory. E.g., for one degree of freedom, we may integrate once and get an (implicit) expression for $x(t)$:

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \;\Rightarrow\; \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)} \;\Rightarrow t - t_0 = \pm\int_{x_0}^{x}\frac{dx'}{\sqrt{\frac{2}{m}\left(E - V(x')\right)}} \tag{7}$$

[1]The E_0-level set of a real valued function $E(x, p)$ is the set of points where the function takes a fixed value E_0. If the set of points is a curve, we call it a level curve. Level curves of the height function over a hilly region are called level contours, and are drawn in maps.

In effect we have solved Newton's second order equation of motion in two steps. Energy is the constant of integration in the first step and x_0 is the second constant of integration. Our answer expresses t as a function of x. We must invert it to find trajectories $x(t)$ with energy E and initial location x_0 at t_0. Interestingly, there is often more than one trajectory with fixed energy and initial location, corresponding to the $\pm$ signs. This is to be expected, since specification of energy allows two possible initial velocities in general $v_0 = v(t_0) = \pm\sqrt{(2/m)(E - V(x_0))}$ (if the particle is at a turning point of the potential $E = V(x_0)$, initially, then $v_0 = 0$ and the particle has only one way to go, 'down hill'). So specification of energy and initial location is, in general, not a complete specification of the instantaneous state of the particle.

• If there is no force, then each of the components of momentum is conserved, since $\dot{\mathbf{p}} = \mathbf{F} = 0$ (this is Newton's first law). If the force only acts downwards, then the horizontal components of momentum p_x, p_y are conserved.

• The angular momentum (or moment of momentum) of a particle about a fixed point (origin) is $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the position vector of the particle from the chosen origin. In components

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \tag{8}$$

• Newton's force law then implies that the rate of change of angular momentum is the torque (or moment of force):

$$\dot{L} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \; = \; \frac{1}{m}\mathbf{p} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} \; = \; \mathbf{r} \times \mathbf{F} \; \equiv \; \vec{\tau} \equiv \mathbf{k}. \tag{9}$$

E.g. For a projectile moving under the vertical gravitational force, the torque must be in the horizontal plane. So the vertical component of angular momentum $L_z = xp_y - yp_x$ must be conserved. Since p_x and p_y are also conserved, we conclude that the trajectory $(x, y, z)(t)$ must be such that its projection on the horizontal plane is a straight line $L_z = xp_y - yp_x$. Of course we knew this and more, the trajectory of a projectile is a parabola over the x-y plane. Again, knowledge of conserved quantities allowed us to clarify the nature of the trajectory.

• The components of position, momentum, angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ and Energy $E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$ are interesting physical quantities associated with the dynamics of a particle. They are examples of *dynamical variables* or observables. In general, a dynamical variable is a smooth real function on phase space. For a single particle dynamical variables may be regarded as functions $f(\mathbf{r}, \mathbf{p})$. The potential $V(\mathbf{r})$ is a function on configuration space and a function on phase space. x^i are called coordinate functions on configuration space. x^i, p_j are called coordinate functions on phase space. In general, dynamical variables change along the trajectory. Conserved quantities are dynamical variables that are constant along *every* trajectory. Of course, the value of a conserved quantity may differ from trajectory to trajectory. For example, energy is a conserved quantity for free particle motion. But the value of energy in general differs from trajectory to trajectory.

3.3 Lagrangian formulation and principle of extremal action

• The principle of extremal action provides a powerful reformulation of Newton's 2nd law, especially for systems with conservative forces. It leads to Lagrange's equations of motion, which are equivalent to Newton's 2nd law. One advantage of Lagrange's equations is that they retain the same form in all systems of coordinates on configuration space.

• The idea of the action principle is as follows. A static solution (time independent trajectory) of Newton's equation for a particle in a potential $m\ddot{x} = -V'(x)$ occurs when the particle is located at an extremum of the potential. The action principle gives a way of identifying (possibly) time-dependent trajectories as extrema of an action function. However, unlike the potential, the action is *not* a function on configuration space. It is a function on the *space of paths* on configuration space, it is called a *functional*. Suppose $q^i(t)$ for $t_i \leq t \leq t_f$ is a path on $\mathcal{Q}$. It is common to use q^i (instead of x^i) for coordinates on configuration space. q^i need not be cartesian coordinates of particles, any system of coordinates will work. Then the action is typically a functional of the form

$$S[q] = \int_{t_i}^{t_f} L(q^i, \dot{q}^i)\, dt. \tag{10}$$

Here $L(q^i, \dot{q}^i)$ is called the Lagrangian of the system, a function of coordinates and velocities. Geometrically, $q^i(t)$ is a path on configuration space. At any instant, $q^i(t)$ is a point on configuration space and $\dot{q}^i(t)$ is a tangent vector to the curve at that point. For a suitable L (usually the difference between kinetic and potential energies, $L = T - V$) Newtonian trajectories are extrema of S.

• In other words, we consider the problem of determining the classical trajectory that a particle must take if it is at q_i at t_i and q_f at t_f. Instead of specifying the initial velocity, we give the initial and final positions at these times. Which among all the paths that connect these points solve Newton's equation? The action (variational) principle says that classical trajectories are extrema of S. Note that unlike the initial value problem for Newton's equations, where $q^j(t_i), \dot{q}^j(t_i)$ are specified, this initial-final value problem (where $q^j(t_i)$ and $q^j(t_f)$) are specified, may not have a unique solution. The action may have more than one extremum. Give an example!

• Aside: Note that specification of initial and final locations (and times) as well as initial velocity, would be an over-specification of the problem. In general, there would be no trajectory that satisfies these conditions. This goes back to the fact that Newton's equations are second order in time, they admit two sets of initial conditions.

• To understand this idea, we need to determine the conditions for S to be extremal. These conditions are called Euler-Lagrange equations. In the static case, the condition for $V(x)$ to be extremal is that its change under an infinitesimal shift δx of x must vanish to first order in δx, this turns out to be the condition $V'(x) = 0$.

• The Euler-Lagrange equations are got by computing the infinitesimal change in action δS under a small change in path $q^i(t) \to q^i(t) + \delta q^i(t)$ while holding the initial and final locations $q^i(t_i), q^i(t_f)$ unchanged. Assuming the variation in the path is such that $\frac{d\delta q(t)}{dt} = \delta\dot{q}$, we get

$$\begin{aligned} \delta S &= \sum_{i=1}^{n} \int_{t_i}^{t_f} dt' \left\{ \frac{\partial L}{\partial q^i} \delta q^i(t') + \frac{\partial L}{\partial \dot{q}^i} \delta\dot{q}^i(t') \right\} + \mathcal{O}(\delta q)^2 \\ &= \int_{t_i}^{t_f} \delta q^i(t') \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt'} \frac{\partial L}{\partial \dot{q}^i} \right) dt' + \delta q^i(t_f) \frac{\partial L}{\partial \dot{q}^i(t_f)} - \delta q^i(t_i) \frac{\partial L}{\partial q^i(t_f)} + \mathcal{O}(\delta q)^2 \end{aligned} \tag{11}$$

We integrated by parts to isolate the coefficient of δq. The last two 'boundary terms' are zero due to the initial and final conditions and so the condition $\delta S = 0$ can be reduced to a condition that must hold at each time, since $\delta q^i(t')$ are arbitrary at each intermediate time. So choosing,

roughly, $\delta q^i(t') = 0$ except for a specific time $t' = t$ we get the Euler-Lagrange (EL) (or just Lagranges's) equations

$$\frac{\partial L}{\partial q^i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i(t)} = 0, \quad i = 1, 2, \ldots n. \tag{12}$$

• Now let us see how the principle of extremal action implies Newton's equation of motion for a particle in a potential, by a suitable choice of L. Comparing $m\ddot{q} = -V'(q)$ with the EL equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ (both now in cartesian coordinates) we notice that if we choose $L = \frac{1}{2}m\dot{q}^2 - V(q)$, then

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \text{and} \quad \frac{\partial L}{\partial q} = -V'(q) \tag{13}$$

and the EL equation reduces to Newton's equation. The Lagrangian scheme has the advantage of generating all the equations of motion from a single function. Moreover, the Euler-Lagrange equations (written in terms of L) may be literally carried over to any coordinate system, so q_i need not be Cartesian coordinates and are often called generalized coordinates. There are as many generalized coordinates q_i as there are degrees of freedom. So for a pair of particles in a room, there would be six generalized coordinates $q_1, \cdots, q_6$.

• The example of a harmonic oscillator (particle connected to a spring of force constant k). Here the restoring force $-kx$ arises from a potential $V(x) = \frac{1}{2}kx^2$, where x is the extension of the spring. So,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{14}$$

and Lagrange's equation $\frac{d(m\dot{x})}{dt} = -kx$ reproduces Newton's equation.

3.4 Non-uniqueness of Lagrangian

• A Lagrangian for a given system of equations is not uniquely defined. For instance, we may add a constant to $L(q, \dot{q}, t)$ without affecting the EL equations, this is like changing the zero of potential energy. We may also multiply the Lagrangian by a constant. Another source of non-uniqueness arises from the freedom to add the total time derivative of any (differentiable) function $F(q, t)$ to the Lagrangian. The change in the action is

$$L_{new} = L_{old} + \dot{F} \;\; \Rightarrow \;\; S_{new} = S_{old} + \int_{t_i}^{t_f} \frac{dF}{dt}dt = S_{old} + F(q(t_f), t_f) - F(q(t_i), t_i) \tag{15}$$

But this quantity involving F on the rhs has zero variation since $t_i, t_f, q(t_i), q(t_f)$ are all held fixed as the path is varied. So $\delta S_{old} = \delta S_{new}$. So the addition of $\dot{F}$ to L does not affect the EL equations. Notice that we could not allow F to depend on $\dot{q}$ since $\delta\dot{q}(t_i), \delta\dot{q}(t_f) \neq 0$ in general and such an F would modify the EL equations. There is no restriction on the initial and final velocities of the perturbed paths.

3.5 Conjugate momentum and cyclic coordinates

• It is important to bear in mind that the Lagrangian $L(q, \dot{q})$ is a function of the coordinates q and velocities $\dot{q}$, and that the momentum p is a derived concept. The momentum p_i conjugate to the coordinate q^i is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{16}$$

In general conjugate momenta do not have the dimensions MLT^{-1}, just as generalized coordinates q^i do not necessarily have dimensions of length. Indeed, an angle coordinate is dimensionless. Conjugate momentum is a useful concept. The momentum p_j conjugate to a coordinate q^j that does not appear in the Lagrangian is automatically conserved.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j(t)} = \frac{\partial L}{\partial q^j(t)} = 0. \tag{17}$$

Such a coordinate is called a cyclic coordinate. For a free particle moving on a line, $L = \frac{1}{2}m\dot{x}^2$ and x is a cyclic coordinate. So its conjugate momentum $p_x = m\dot{x}$ is conserved $\dot{p}_x = 0$.

• Not every conserved quantity may arise as the momentum conjugate to a cyclic coordinate. For example, if we use cartesian coordinates for the particle in a central potential on a plane, $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2+y^2})$, then neither coordinate is cyclic and neither of the momenta ($p_x = m\dot{x}, p_y = m\dot{y}$) are conserved. But as we see below the momentum conjugate to the cyclic angular coordinate is conserved. So some physical insight/cleverness/luck may be needed in choosing coordinate systems in which one or more coordinate is cyclic.

• For a particle moving on a plane, in polar coordinates, $x = r\cos\phi$ and $y = r\sin\phi$. Then the components of velocity are

$$v = (\dot{x}, \dot{y}) \quad \text{where} \quad \dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}, \qquad \dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}. \tag{18}$$

So $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$ and the Lagrangian for a central potential $V(r)$ is

$$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \tag{19}$$

The momenta conjugate to (r, ϕ) are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \tag{20}$$

They coincide with the radial component of linear momentum and the z component of angular momentum. Moreover, the first of Lagrange's equations is

$$\dot{p}_r = m\ddot{r} = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 - V'(r). \tag{21}$$

This is the balance of radial acceleration, centripetal 'force' and central force. On the other hand,

$$\dot{p}_\phi = \frac{d}{dt}(mr^2\dot{\phi}) = \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad mr\ddot{\phi} = -2m\dot{r}\dot{\phi}. \tag{22}$$

This states the conservation of angular momentum, and involves the so-called Coriolis term on the rhs when written out. Note that Newton's equations do not take the same form in all systems of coordinates. There is no force in the $\hat{\phi}$ direction, yet the naive 'angular acceleration' $m\ddot{\phi}$ is non-zero. On the other hand, Lagrange's equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$ are valid in all systems of coordinates. So q could be a Cartesian or polar coordinate for instance.

3.6 Hamiltonian and its conservation

• Besides the momenta conjugate to cyclic coordinates, the Lagrangian formulation leads automatically to another conserved quantity, the hamiltonian. For a moment suppose the Lagrangian depends explicitly on time $L = L(q(t), \dot{q}(t), t)$. Then

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t} = \dot{p}\dot{q} + p\ddot{q} + \frac{\partial L}{\partial t} = \frac{d(p\dot{q})}{dt} + \frac{\partial L}{\partial t} \quad \Rightarrow \quad \frac{d(p\dot{q} - L)}{dt} = -\frac{\partial L}{\partial t}. \tag{23}$$

So if we define the hamiltonian $H = p\dot{q} - L$, then $\dot{H} = -\frac{\partial L}{\partial t}$. So if the Lagrangian does not depend explicitly on time, then H is conserved.

• For many of the systems we study, the hamiltonian coincides with energy $E = T + V$. This is always the case if the Lagrangian $L = T - V$ is such that the kinetic energy is quadratic in velocities and the potential energy $V(q)$ depends only on coordinates. For example, if $L = \frac{1}{2}m\sum_i \dot{q}_i^2 - V(q)$, then the conjugate momenta are $p_i = m\dot{q}_i$ and

$$H = p_i\dot{q}_i - L = m\sum_i \dot{q}_i^2 - \frac{1}{2}m\sum_i \dot{q}_i^2 + V(q) = \frac{1}{2}m\dot{q}_i^2 + V(q) = T + V. \tag{24}$$

3.7 From symmetries to conserved quantities: Noether's theorem on invariant variational principles

• Newton/Lagrange equations of classical mechanics have been formulated as conditions for the action $S = \int L\,dt$ to be extremal. Many concepts (such as symmetries) may be formulated more simply in terms of the action/Lagrangian than in terms of the equations of motion.

• If a coordinate q^j is absent in the Lagrangian (q^j *is a cyclic coordinate*), then the corresponding conjugate momentum $p_j = \frac{\partial L}{\partial \dot{q}^j}$ is conserved in time. This follows from Lagrange's equation $\dot{p}_j = \frac{\partial L}{\partial q^j}$. If the Lagrangian is independent of a coordinate, then in particular, it is unchanged when this coordinate is varied $\delta L = 0$ under $q^j \to q^j + \delta q^j$. We say that translations of q^j are a symmetry of the Lagrangian. This relation between symmetries and conserved quantities is deeper, it goes beyond mere translations of a coordinate.

• A transformation of coordinates $q^i \to \tilde{q}^i$ is a symmetry of the equations of motion (eom) if it leaves them unaltered: i.e., the eom for $\tilde{q}$ is the same as that for q. Symmetries usually allow us to produce new solutions from known ones. For example, the free particle equation $m\ddot{q} = 0$ is left unchanged by a translation of the coordinate $q \to \tilde{q} = q + a$ for any constant length a. Now $q = 0$ is one static solution. We may use the symmetry under translations to produce other static solutions, namely $q = a$ for any a, i.e., the particle is at rest at location with coordinate a rather than at the origin. Incidentally, the momentum of a free particle is conserved in time. We will see that such symmetries are associated with conserved quantities. On the other hand, the equation of motion of a particle attached to a spring $m\ddot{q} = -kq$ is non-trivially modified by a translation of the coordinate $q \to \tilde{q} = q + a$ since $\tilde{q}$ satisfies a different equation $m\ddot{\tilde{q}} = -k\tilde{q} + ka$. Moreover, $p = m\dot{q}$ is not (in general) conserved for a particle executing simple harmonic motion, the momentum is zero at the turning points and maximal at the point of equilibrium.

• It is important to note that not every transformation of q qualifies as a symmetry of the equations of motion. We have already argued that every transformation of coordinates leaves

the form of Lagrange's equations invariant. So here, when we say 'leaves the eom invariant', we aren't referring to the form of Lagranges equations i.e., $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ but to the differential equations written out explicitly (without any Lagrange function present).

• A symmetry of the Lagrangian is a transformation that leaves L unchanged. E.g. the free particle $L = \frac{1}{2}m\dot{q}^2$ is unchanged under the shift $q \to q + a$. It follows that the action $S[q] = \int_{t_1}^{t_2} \frac{1}{2}m\dot{q}^2\, dt$ is unchanged under the shift $q \to q + a$. Since the eom are the conditions for S to be stationary, a symmetry of the Lagrangian must also be a symmetry of Lagrange's equations. Noether's theorem constructs a conserved quantity associated to each infinitesimal symmetry of the Lagrangian[2]. Let us see how. Suppose the infinitesimal change $q^i \to q^i + \delta q^i$ leaves the Lagrangian unchanged to linear order in δq. Then it is automatically an infinitesimal symmetry of the action. Let us explicitly calculate the first variation of the action for paths between the times t_1 and t_2, $S[q + \delta q] = S[q] + \delta S[q]$. Up to terms of order $(\delta q)^2$ we get

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left[\delta q^i \frac{\partial L}{\partial q^i} + \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}\right] dt = \int_{t_1}^{t_2} \left[\delta q^i \frac{\partial L}{\partial q^i} + \frac{d}{dt}\left(\delta q^i \frac{\partial L}{\partial \dot{q}^i}\right) - \delta q^i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right] dt \\ &= \delta q^i(t_2)\frac{\partial L}{\partial \dot{q}^i}(t_2) - \delta q^i(t_2)\frac{\partial L}{\partial \dot{q}^i}(t_1) + \int_{t_1}^{t_2} \delta q^i \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right] dt \end{aligned} \tag{25}$$

So far, this is true for any path and for any infinitesimal change δq^i. Let us now specialize to infinitesimal changes about a trajectory, so that $q^i(t)$ satisfies Lagrange's equations and the last term vanishes. Further more, we assume that the transformation is an infinitesimal symmetry of the Lagrangian, so that $\delta S = 0$:

$$0 = \delta S = \delta q^i(t_2)\frac{\partial L}{\partial \dot{q}^i}(t_2) - \delta q^i(t_2)\frac{\partial L}{\partial \dot{q}^i}(t_1). \tag{26}$$

Since t_1, t_2 are arbitrary, the quantity $\delta q^i \frac{\partial L}{\partial \dot{q}^i}$ must be constant along a trajectory. In other words, an infinitesimal symmetry $q \to q + \delta q$ of the Lagrangian implies that the quantity $Q = p_i(t)\delta q^i(t) = \vec{p}\cdot\delta\vec{q}$ is a constant of the motion, i.e. the dynamical variable Q has the same value at all points along a trajectory. Q is called a Noether conserved 'charge' by analogy with the conservation of electric charge.

• E.g. 1: We already saw that the free particle Lagrangian is translation invariant with $\delta q^i = a^i$ where a^i are the components of an arbitrary infinitesimal vector. It follows that $Q = a^i p_i = \vec{p}\cdot\vec{a}$ is a conserved quantity. In other words, the component of momentum in *any* direction is conserved.

• E.g. 2: Now consider a particle in a central potential $V(\mathbf{q}^2)$ so that the Lagrangian is

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} - V(\mathbf{q}\cdot\mathbf{q}) \tag{27}$$

Let us first show that L is invariant under rotations of three dimensional space $\vec{q} \to R\vec{q}$ where R is any (special) orthogonal rotation matrix ($R^tR = I, \det R = 1$). Recall that the dot product is defined as $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^t\mathbf{b}$ for any column vectors $\mathbf{a}, \mathbf{b}$ and that $(R\mathbf{a})^t = \mathbf{a}^tR^t$ for any matrix R and ^{t} denotes transposition. Thus

$$L(R\mathbf{q}, R\dot{\mathbf{q}}) = \frac{1}{2}m\dot{\mathbf{q}}R^tR\dot{\mathbf{q}} - V(\mathbf{q}^tR^tR\mathbf{q}) = \frac{1}{2}m\dot{\mathbf{q}}^t\dot{\mathbf{q}} - V(\mathbf{q}^t\mathbf{q}) = L(\mathbf{q}, \dot{\mathbf{q}}). \tag{28}$$

[2]There is a generalization to the case where the Lagrangian changes by a total time derivative.

So the Lagrangian is invariant under rotations. Noether's theorem, however, refers to infinitesimal transformations, rotations in this case. So let us find a formula for an infinitesimal rotation.

• Suppose we make an infinitesimal rotation of the vector $\mathbf{q}$ about the axis $\hat{n}$ by a small angle θ counter-clockwise. Then the vector $\mathbf{q}$ sweeps out a sector of a cone. Suppose $\mathbf{q}$ makes an angle ϕ with respect $\hat{n}$, so that the opening angle of the cone is ϕ. Then the rotated vector $\tilde{\mathbf{q}}$ also makes an angle ϕ with respect to the axis $\hat{n}$. Let $\delta\mathbf{q} = \tilde{\mathbf{q}} - \mathbf{q}$ be the infinitesimal change in $\mathbf{q}$. By looking at the base of this cone, we find that it is a sector of a circle with radius $q\sin\phi$ and opening angle θ. So we find that $|\delta\mathbf{q}| = \theta q \sin\phi$. Moreover δq points in the direction of $\hat{n}\times\mathbf{q}$. Thus, under a counter-clockwise rotation about the axis $\hat{n}$ by a small angle θ, the change in $\mathbf{q}$ is

$$\delta\mathbf{q} = \theta\,\hat{n}\times\mathbf{q} \qquad \text{and} \qquad \delta\dot{\mathbf{q}} = \theta\,\hat{n}\times\dot{\mathbf{q}} \tag{29}$$

In particular, we see that $\delta\mathbf{q}$ and $\delta\dot{\mathbf{q}}$ are orthogonal to $\mathbf{q}$ and $\dot{\mathbf{q}}$ respectively.

• Now let us check that the Lagrangian is invariant under infinitesimal rotations:

$$L(\mathbf{q}+\delta\mathbf{q},\dot{\mathbf{q}}+\delta\dot{\mathbf{q}}) \approx \frac{1}{2}m\dot{\mathbf{q}}^2 + \frac{1}{2}m\dot{\mathbf{q}}\cdot\delta\dot{\mathbf{q}} + \frac{1}{2}m\delta\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} - V(\mathbf{q}^2 + \mathbf{q}\cdot\delta\mathbf{q} + \delta\mathbf{q}\cdot\mathbf{q}) = L(\mathbf{q},\dot{\mathbf{q}}) \tag{30}$$

The last equality follows on account of the orthogonality properties just mentioned. Thus the Lagrangian (and action) are invariant under infinitesimal rotations. The resulting conserved quantity from Noether's theorem is

$$Q = \vec{p}\cdot\theta\ (\hat{n}\times\vec{q}) = \theta\,\hat{n}\cdot(\vec{q}\times\vec{p}) = \theta\,\vec{L}\cdot\hat{n}. \tag{31}$$

Since Q is conserved for any small angle θ and for any axis of rotation $\hat{n}$, we conclude that the component of angular momentum in *any* direction is conserved. So the angular momentum vector is a constant of motion $\frac{d\vec{L}}{dt} = 0$, a fact we are familiar with from the Kepler problem for the $1/r$ central potential. We also knew this since the torque $\vec{r}\times\vec{F}$ on such a particle about the force centre vanishes: the moment arm and force both point radially.

3.8 Coordinate invariance of the form of Lagrange's equations

• Lagrange's equations $\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} = \frac{\partial L}{\partial q_i}$ take the same form in all systems of coordinates on $\mathcal{Q}$. This is because in deriving them from the action variational principle, q^i could be any coordinates, we did not assume that they were Cartesian.

• Let us further illustrate the coordinate invariance of the form of Lagrange's equations and the non-invariance of the form of Newton's equations. Consider a free particle on the positive half line $q > 0$ with Lagrangian $L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2$. In this case Lagrange's equation reduces to $\ddot{q} = 0$. Now let us choose a different coordinate system on configuration space, defined by $Q = q^2$. If Newton's equation $\mathbf{F} = m\ddot{\mathbf{r}}$ were coordinate invariant then we would guess that the equation of motion for Q must be $m\ddot{Q} = 0$ since there is no force. But this is not the correct equation of motion. The correct equation of motion may be obtained by making the change of variable $q \to Q$ in $\ddot{q} = 0$. Using $\dot{Q} = 2q\dot{q}$ and $\ddot{Q} = 2q\ddot{q} + 2\dot{q}^2$ one arrives at

$$2Q\ddot{Q} - \dot{Q}^2 = 0. \tag{32}$$

This is the equation of motion written in terms of Q. Notice that it *doesn't* have the same form as Newton's equation for q.

• On the other hand, let us find the Lagrangian as a function of Q and $\dot{Q}$ and the resulting Lagrange equations to see if they give the correct result found above. First we express the Lagrange function in terms of the new coordinate

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 = \frac{m\dot{Q}^2}{8Q} = \tilde{L}(Q,\dot{Q}). \tag{33}$$

If the form of Lagrange's equations are the same in the Q coordinate system we must have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{\partial L}{\partial Q} \quad \text{or} \quad \frac{m\ddot{Q}}{4Q} - \frac{m\dot{Q}^2}{4Q^2} = -\frac{m\dot{Q}^2}{8Q^2} \tag{34}$$

Simplifying, we see that Lagrange's equation agrees with the transformed version of Newton's equation $2Q\ddot{Q} - \dot{Q}^2 = 0$. So we verified that Lagrange's equations take the same form in both the q and Q coordinates. As mentioned above, this is generally true for any choice of coordinates on configuration space.

• It may be noted that the differential equations $\ddot{q} = 0$ and $\frac{2\ddot{Q}}{Q} = \frac{\dot{Q}^2}{Q^2}$ are *not of the same form*, though they are *equivalent*. What we found is that the eom, *when expressed in terms of the respective Lagrange functions* take the same form: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ and $\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}} = \frac{\partial \tilde{L}}{\partial Q}$.

4 Hamiltonian formalism of mechanics

4.1 Hamilton's equations

• We introduced the hamiltonian $H = p_i\dot{q}^i - L(q,\dot{q})$ as an interesting conserved quantity implied by Lagrange's equations. Here $p_i = \frac{\partial L}{\partial \dot{q}^i}$. To understand H better, let us compute its differential using Lagrange's equations

$$dH = p_i d\dot{q}^i + \dot{q}^i dp_i - \frac{\partial L}{\partial q^i}dq^i - \frac{\partial L}{\partial \dot{q}^i}d\dot{q}^i = p_i d\dot{q}^i + \dot{q}^i dp_i - \dot{p}_i dq^i - p_i d\dot{q}^i = -\dot{p}_i dq^i + \dot{q}^i dp_i \tag{35}$$

This reveals that the independent variables in H are the generalized coordinates q^i and the generalized momenta p_i, the terms involving the differentials of velocities cancelled out. So we should think of H as $H(q,p)$. Now by the definition of partial derivatives,

$$dH = \frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i. \tag{36}$$

Comparing, we find that the time derivatives of coordinates and momenta may be expressed in terms of partial derivatives of the Hamiltonian

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{37}$$

• These are a system of $2n$ first order ordinary differential equations, for a system with n degrees of freedom. They are called Hamilton's equations. There are twice as many of them compared to Newton's or Lagrange's equations, which are second order in time. Hamilton's equations treat position and momentum on a more equal footing, except for a sign. They give us yet another way of expressing the equations of time evolution.

• We regard the hamiltonian $H(q,p)$ as a function on phase space, i.e., as a function of positions and momenta (the current state of the system) rather than positions and velocities. The solution of Hamilton's equations give $q(t)$ and $p(t)$, i.e., the trajectory on phase space. To find it we need to specify the initial state of the system i.e., $q(0)$ and $p(0)$.

• To express H as a function on phase space, we must express $H = p_i\dot{q}^i - L(q^i, \dot{q}^i)$ as a function of q^j and p_j. This is done by eliminating velocities $\dot{q}^i$ in favor of q, p using the definition of conjugate momenta $p_j = \frac{\partial L}{\partial \dot{q}^j}$.

• E.g. particle in a 1D potential. Then $L = \frac{1}{2}m\dot{q}^2 - V(q)$ and $p = m\dot{q}$ so $\dot{q} = p/m$. Then $H = p\dot{q} - L = pp/m - p^2/2m + V(q) = p^2/2m + V(q)$. Hamilton's equations are $\dot{q} = \frac{\partial H}{\partial p} = p/m$ which recovers the definition of conjugate momentum and $\dot{p} = -\frac{\partial H}{\partial q} = -V'(q)$ which is Newton's second law.

• If the Lagrangian is explicitly time dependent, then the hamiltonian is not conserved. Even so, the eq. of motion may be expressed in terms of the hamiltonian. The differential of the hamiltonian is

$$dH = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t}dt \quad \text{and} \quad dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt. \tag{38}$$

Comparing, we get

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{39}$$

So even for a time-dependent H, hamilton's equations for coordinates and momenta take the same form.

• One of the main advantages of the Hamiltonian formalism (like the Lagrangian one) over the Newtonian one is that Hamilton's equations take the same form in all systems of coordinates on configuration space. By contrast, Newton's equations $m\ddot{x}_i = F_i$ look quite different in curvilinear coordinates, as we have seen. Let us illustrate this assertion with the example of a particle free to move on a plane in a central potential. Newton's equations are then $m\ddot{x} = -\frac{\partial V}{\partial x}$ and $m\ddot{y} = -\frac{\partial V}{\partial y}$ and the conserved energy is $E = T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + V(\sqrt{x^2+y^2})$. The coordinates are x, y and the Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2+y^2})$. The conjugate momenta $p_x = m\dot{x}$ and $p_y = m\dot{y}$ are just the components of linear momentum in the x, y directions.

• Let us transform to polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$. Then the components of velocity are

$$v = (\dot{x}, \dot{y}) \quad \text{where} \quad \dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}, \qquad \dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}. \tag{40}$$

So $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$ and the Lagrangian is

$$L = T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r). \tag{41}$$

The momenta conjugate to (r, ϕ) are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \tag{42}$$

They coincide with the radial component of linear momentum and the 'z-component' of angular momentum, as we have seen in the assignment.

• Though we call p_ϕ the momentum conjugate to ϕ, it does not have dimensions of momentum, it actually has dimensions of angular momentum. In the same way we refer to r, ϕ as coordinates, though they don't both have dimensions of length. But in general the product a coordinate and its conjugate momentum always has dimensions of action or angular momentum.

• The hamiltonian is now expressed in polar coordinates and momenta by eliminating velocities

$$H = p_\phi \dot{\phi} + p_r \dot{r} - L = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r). \tag{43}$$

• Now Newton's equations $m\ddot{x} = -\partial_x V; m\ddot{y} = -\partial_y V$ can be expressed in polar coordinates, indeed we had found

$$m\ddot{r} = mr\dot{\phi}^2 - V'(r) \quad \text{and} \quad mr\ddot{\phi} = -2m\dot{r}\dot{\phi}. \tag{44}$$

Notice that Newton's equations in polar coordinates have a different form than in cartesian coordinates. If we had naively regarded $m\ddot{r}$ as an acceleration and observed that there is no force on the particle, we would have got a wrong equation. Though there is no force on the particle, neither $\ddot{r}$ nor $\ddot{\phi}$ is zero. Of course, we sometimes say that $mr\dot{\phi}^2$ is a centripetal force, but this is not an external force like gravity, it is just another term in the acceleration due to the curvilinear coordinates.

• On the other hand, let us write down Hamilton's equations, which we claimed take the same form in all coordinate systems

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}, \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0, \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - V'(r). \tag{45}$$

• Let us check if these equations agree with Newton's equations. The first two of Hamilton's equations just reproduce the definitions of p_r and p_ϕ. The third says that the angular momentum $p_\phi = L_z$ is conserved. The last one along with the first two is equivalent to Newton's equation $m\ddot{r} = mr\dot{\phi}^2 - V'(r)$ for the balance of radial acceleration, centrifugal acceleration and central force.

• The coordinate ϕ does not appear in the hamiltonian, it is a cyclic coordinate. So the conjugate momentum (angular momentum p_ϕ) is conserved. This was not quite obvious from Newton's equations, though we knew it. In general the momentum p conjugate to any cyclic coordinate q is conserved since $\dot{p} = -\frac{\partial H}{\partial q} = 0$.

4.2 Hamiltonian from Legendre transform of Lagrangian

• The Legendre transform gives a way of summarizing the passage from Lagrangian to Hamiltonian. Notice that the definition of conjugate momentum $p = \frac{\partial L}{\partial \dot{q}}$ is the condition for $p\dot{q} - L$ to be extremal with respect to small variations in $\dot{q}$. Moreover, the extremal value of this function is the hamiltonian $H(q, p)$. Thus, we may write

$$H(q, p) = \text{ext}_{\dot{q}} \left[p_i \dot{q}^i - L(q, \dot{q}) \right] \tag{46}$$

The extremization is carried out with respect to all the generalized velocities. The key step in the Legendre transform is to solve for the velocities in terms of the momenta and coordinates using $p_i = \frac{\partial L}{\partial \dot{q}^i}$. This is not always possible. It could happen that there is none or more than one solution $\dot{q}$ for given q, p. Then H would not be a single-valued function of coordinates and momenta. A condition that guarantees that the Legendre transform exists as a single-valued function is convexity (or concavity) of the Lagrangian as a twice differentiable function of $\dot{q}$. L is convex provided the 2^{nd} derivative $\frac{\partial^2 L}{\partial \dot{q}^2}$ is positive everywhere on configuration space $\mathcal{Q}$. This condition is satisfied by $L = \frac{1}{2}m\dot{q}^2$ if $m > 0$. Here $p\dot{q} - L = p\dot{q} - \frac{1}{2}m\dot{q}^2$ is a quadratic function of $\dot{q}$ and has a unique extremum (in fact a maximum) for any p.

• On the other hand, let us attempt to compute the Legendre transform of $L = \frac{1}{4}\dot{q}^4 - \frac{1}{2}\dot{q}^2$. We expect to run into trouble. Indeed, there is often more than one solution (typically 1, 2 or 3) $\dot{q}$ for a given p when we try to solve for $\dot{q}$ in $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}^3 - \dot{q}$. In this case, the Legendre transform H is not single-valued on some parts of phase space (a range of momenta around zero).

• When the Legendre transform is defined, the Lagrangian can be re-obtained from $H(q, p)$ by an (inverse) Legendre transform

$$L(q, \dot{q}) = \text{ext}_p \left[p\dot{q} - H(q, p) \right]. \tag{47}$$

How do we get this? Recall the definition $H(q, p) = p\dot{q} - L(q, \dot{q})$, so $L(q, \dot{q}) = p\dot{q} - H(q, p)$. Now on the rhs, we see three variables $p, q, \dot{q}$ while on the lhs we only have $q, \dot{q}$. So we must say how to express $\dot{q}$ in terms of q, p. This is provided by the first of Hamilton's equations $\dot{q} = \frac{\partial H(q,p)}{\partial p}$. Now we notice that this first Hamilton equation is the condition for $p\dot{q} - H$ to be extremal in p. Moreover, the value of this function at its extremum is just the Lagrangian. Thus we have $L(q, \dot{q}) = \text{ext}_p \left[p\dot{q} - H(q, p) \right]$.

4.3 Non-uniqueness of Hamiltonian

Just as there are many Lagrangians whose Euler-Lagrange equations give the same Newton's second law equation, there are in general many hamiltonians that lead to the same 2nd order equation for the coordinates. Of course, the hamiltonian is defined up to an additive constant. But there is further freedom. Recall that we could add the total time derivative of a function of coordinates and time to the Lagrangian, without changing the dynamics. Simply Legendre transforming this modified Lagrangian will lead to a modified Hamiltonian which corresponds the same Newton equation. For example, consider a free particle in 1d. For any differentiable function $b(q)$ show that the EL equation for $L = \frac{1}{2}m\dot{q}^2 + b(q)\dot{q}$ is simply $m\ddot{q} = 0$. The corresponding conjugate momentum is $p = m\dot{q} + b(q)$. The Hamiltonian after Legendre transformation is $H = (p - b(q))^2/2m$. Show that Hamilton's equations $\dot{q} = (p - b(q))/m$ and $\dot{p} = b'(q)(p - b(q))/m$ reduce to the free particle equation of motion $m\ddot{q} = 0$ for any differentiable function $b(q)$.

4.4 Poisson brackets

• Consider a particle (or system of particles) with configuration space R^n with generalized coordinates q_i and generalized momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$. To motivate the idea of Poisson brackets, let us use Hamilton's equations ($\dot{q}^i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q^i}$) to find the time evolution of any

dynamical variable $f(q,p;t)$. f is in general a function on phase space, which could depend explicitly on time.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}\right) = \frac{\partial f}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i}\right) = \frac{\partial f}{\partial t} + \{f,H\}. \quad (48)$$

Here we introduced Poisson's bracket of f with the hamiltonian. More generally, the p.b. of two dynamical variables evaluated at the same time t gives another dynamical variable at the same time, defined as[3]

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right). \quad (49)$$

So the time derivative of any observable is given by its Poisson bracket with the hamiltonian (aside from any explicit time-dependence). From here on, we will restrict to observables that are not explicitly time-dependent (i.e. depend on time only via $q(t)$ and $p(t)$), unless otherwise stated. Hamilton's equations for time evolution may now be written

$$\dot{q}^i = \{q^i, H\} \quad \text{and} \quad \dot{p}_j = \{p_j, H\}. \quad (50)$$

If H isn't explicitly dependent on time, then time does not appear explicitly on the RHS of hamilton's equations. In this case, we say that the ODEs for q and p are an autonomous system.

• One advantage of Poisson brackets is that the time evolution of any observable $f(q,p)$ is given by an equation of the same sort $\dot{f} = \{f,H\}$. We say that the hamiltonian generates infinitesimal time evolution via the Poisson bracket, since $f(t+\delta t) \approx f(t) + (\delta t)\{f,H\}$.

$$f(q(t+\delta t), p(t+\delta t)) = f(q(t),p(t)) + (\delta t)\ \{f,H\} + \mathcal{O}((\delta t)^2). \quad (51)$$

• If $\{f,g\} = 0$ we say that f 'Poisson commutes' with g. In particular, f is a constant of motion iff it Poisson commutes with the hamiltonian, $\dot{f} = 0 \Leftrightarrow \{f,H\} = 0$. We begin to see the utility of the Poisson bracket in the study of conserved quantities.

• The Poisson bracket has some notable properties. The p.b. of any dynamical variable with a constant is zero. The Poisson bracket is **linear** in each entry. Verify that $\{f,cg\} = c\{f,g\}$ and $\{f,g+h\} = \{f,g\} + \{f,h\}$ etc. where c is a real constant.

• The Poisson bracket is **anti-symmetric** in the dynamical variables $\{f,g\} = -\{g,f\}$. In particular, the p.b. of any observable with itself vanishes $\{h,h\} = 0$. A special case of this encodes the conservation of energy. Assuming H isn't explicitly dependent on time,

$$\frac{dH}{dt} = \{H,H\} = 0. \quad (52)$$

• Since the above formula for the p.b. involves only first order derivatives of f, the p.b. satisfies the **Leibnitz/product rule** of differential calculus. Check that

$$\{fg,h\} = f\{g,h\} + \{f,h\}g \quad \text{and} \quad \{f,gh\} = \{f,g\}h + g\{f,h\}. \quad (53)$$

In the Poisson bracket $\{f,g\}$ we refer to f as the function in the first slot or entry and g as occupying the second. Anti-symmetry ensures that the Leibnitz rule applies to the second entry as well. We say that the p.b. is a **derivation** in either entry.

[3] Some authors (e.g. Landau & Lifshitz) define the p.b. with an overall minus sign relative to our definition.

• The **fundamental Poisson brackets** are between the basic dynamical variables, namely coordinates and momenta. The above formulae give for one degree of freedom

$$\{q,p\}=1 \quad \text{or} \quad \{p,q\}=-1, \quad \text{while} \quad \{q,q\}=0 \quad \text{and} \quad \{p,p\}=0. \tag{54}$$

The last two equations are in fact trivial consequences of the anti-symmetry of the p.b. For n-degrees of freedom we have the fundamental p.b. among the coordinates and momenta

$$\{q^i,p_j\}=\delta^i_j, \quad \text{and} \quad \{q^i,q^j\}=\{p_i,p_j\}=0 \quad \text{for } 1\leq i,j\leq n\,. \tag{55}$$

These are sometimes called the canonical ('standard') equal-time Poisson bracket relations between coordinates and conjugate momenta. The noun *canon* and the adjective *canonical* refer to something that is standard or conventional.

• **Poisson tensor:** It is convenient to introduce a compact notation for the fundamental p.b. between coordinates and momenta. These may be encoded in the Poisson tensor r^{ij}. Let us combine the coordinates and momenta into a $2n$-component 'grand' coordinate ξ on phase space. We regard ξ as a coordinate on *phase space* and write its components with upper indices:

$$\vec{\xi}=(\xi^1,\xi^2\cdots,\xi^n,\xi^{n+1},\cdots,\xi^{2n})=(\vec{q},\vec{p})=(q^1,\cdots,q^n,p_1,\cdots,p_n) \tag{56}$$

Then check the fundamental Poisson bracket relations may be expressed in terms of ξ^i

$$\{\xi^i,\xi^j\}=r^{ij} \quad \text{where} \quad r^{\text{row column}}=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}. \tag{57}$$

Here r is a $2n\times 2n$ block matrix with $n\times n$ blocks consisting of the identity and zero matrices as indicated. The constant matrix r^{ij} is sometimes called the Poisson 'tensor' of the canonical p.b. relations.

• The p.b. of any pair of observables may now be written in terms of the 'fundamental' p.b. between coordinates and momenta. Show that

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right)=\sum_{i,j=1}^{2n}\frac{\partial f}{\partial \xi^i}\frac{\partial g}{\partial \xi^j}\{\xi^i,\xi^j\}=\sum_{i,j=1}^{2n}r^{ij}\partial_i f\partial_j g. \tag{58}$$

Here $\partial_i=\frac{\partial}{\partial\xi^i}$. Various properties of the canonical Poisson brackets are encoded in the Poisson tensor. Of particular importance to us is the anti-symmetry of r^{ij} (equivalent to antisymmetry of the p.b.) and the constancy of the components r^{ij}.

• **Poisson's theorem:** Perhaps the most remarkable feature of the Poisson bracket is that it can be used to produce new conserved quantities from a pair of existing ones. Poisson's theorem states that if f and g are conserved, then so is $\{f,g\}$. Let us first illustrate this with a couple of examples. For a free particle moving on a plane we know that p_x and p_y are both conserved. Their Poisson bracket is $\{p_x,p_y\}=0$, which is of course a trivially conserved quantity. As a second example, consider a particle moving in three dimensions under the influence of a central potential. We know that $L_x=yp_z-zp_y$ and $L_y=zp_x-xp_z$ are both conserved. We compute $\{L_x,L_y\}$ by using bi-linearity, the Leibnitz rule and other properties of the p.b. and find $\{L_x,L_y\}=L_z$. And indeed, we know that L_z is also a conserved quantity. Similarly we check that

$$\{L_x,L_y\}=L_z, \quad \{L_y,L_z\}=L_x \quad \text{and} \quad \{L_z,L_z\}=L_y. \tag{59}$$

• **Jacobi identity:** More generally, Poisson's theorem is a consequence of the Jacobi identity. For any three dynamical variables f, g and h, the following cyclic sum of 'double' Poisson brackets vanishes:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{60}$$

Using anti-symmetry we could write the Jacobi identity also as

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \tag{61}$$

Before we prove the Jacobi identity, let us use it to establish Poisson's theorem. Suppose f, g are conserved so that each of them Poisson commutes with the hamiltonian $h = H$, i.e., $\{f, h\} = \{g, h\} = 0$. Then the Jacobi identity implies that

$$\{\{f, g\}, H\} = 0 \quad \Rightarrow \quad \frac{d}{dt}\{f, g\} = 0. \tag{62}$$

So the p.b. of two conserved quantities is again a conserved quantity.

• **Proof of Jacobi identity:** Let us now prove the Jacobi identity. We wish to evaluate the cyclic sum

$$J = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}. \tag{63}$$

We use the Poisson tensor and the Leibnitz rule to write the first term of J as

$$\{\{f, g\}, h\} = \left(f_i g_j r^{ij}\right)_k h_l r^{kl} = \left[f_{ik} g_j h_l + f_i g_{jk} h_l\right] r^{ij} r^{kl} \tag{64}$$

Here we used subscripts on f, g to denote partial differentiation, $\frac{\partial f}{\partial \xi^i} \equiv f_i$. Adding its cyclic permutations,

$$J = \left[f_{ik} g_j h_l + f_i g_{jk} h_l + g_{ik} h_j f_l + g_i h_{jk} f_l + h_{ik} f_j g_l + h_i f_{jk} g_l\right] r^{ij} r^{kl}. \tag{65}$$

If J has to vanish for any smooth functions f, g, h on phase space, then the terms involving 2nd derivatives of f must mutually cancel as must those involving 2nd derivatives of g or h. So let us consider the two terms involving second derivatives of f, and call the sum J_f. We find

$$\begin{aligned} J_f &= f_{ik} g_j h_l r^{ij} r^{kl} + f_{jk} g_l h_i r^{ij} r^{kl} = f_{ik} g_j h_l r^{ij} r^{kl} + f_{ik} g_l h_j r^{ji} r^{kl} \\ &= f_{ik} g_j h_l r^{ij} r^{kl} + f_{ik} g_j h_l r^{li} r^{kj} = f_{ik} g_j h_l r^{ij} r^{kl} + f_{ki} g_j h_l r^{lk} r^{ij} \\ &= f_{ik} g_j h_l r^{ij} r^{kl} - f_{ik} g_j h_l r^{ij} r^{kl} = 0. \end{aligned} \tag{66}$$

We relabeled indices of summation $i \leftrightarrow j$, $j \leftrightarrow l$ and $i \leftrightarrow k$ in the three successive equalities and finally used the equality of mixed partial derivatives $\frac{\partial^2 f}{\partial \xi^i \xi^k} = \frac{\partial^2 f}{\partial \xi^k \xi^i}$ (variously called Young's or Schwarz' or Clairaut's Theorem) and antisymmetry of the Poisson tensor $r^{kl} = -r^{lk}$. Thus we have shown that $J_f = 0$ and by cyclic symmetry, $J_g = J_h = 0$. Thus $J = 0$ and the Jacobi identity has been established. As a corollary we obtain Poisson's theorem on conservation of p.b. of conserved quantities.

• **Unequal-time Poisson brackets:** It may be noted that unequal-time Poisson brackets contain dynamical information and depend on the hamiltonian. Equal-time Poisson brackets do not depend on the hamiltonian and are in a sense kinematical. Unequal time p.b. $\{f(q(0), p(0)), g(q(t), p(t))\}$ may be reduced to equal time p.b. by solving the equations of motion and expressing $g(q(t), p(t))$ in terms of initial values $q(0)$ and $p(0)$. For example, find $\{q(0), q(t)\}$ for a free particle moving on a line.

5 Oscillations

5.1 Double pendulum: Lagrangian and Hamiltonian

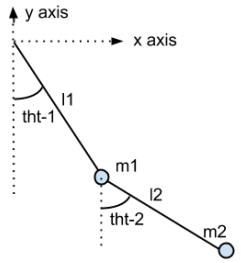


Figure 1: Double pendulum with two bobs of masses m_1, m_2 suspended from a fixed support with massless rods of length l_1, l_2. The respective counterclockwise deflection angles are θ_1, θ_2.

• The double pendulum is a system with a minimal number of degrees of freedom that displays both regular and chaotic dynamics in various energy regimes. It is an interesting and non-trivial model system to study. As a general rule of thumb, if a system admits more conserved quantities, then the dynamics is more constrained and may display more regularity (the best possibility is integrability). A system with 2 degrees of freedom has a 4d phase space. In the absence of any conserved quantity, the trajectory could explore the whole of phase space. If energy is conserved, the trajectory must lie on a 3d constant energy sub-manifold of phase space, determined by initial conditions. If there is another conserved quantity Q functionally independent of energy, then trajectories must lie on the intersection of a constant E and constant Q sub-manifold, which is in general a 2D surface in phase space. We see that the presence of more conserved quantities restricts the dynamics.

• We consider a double pendulum with 'lower' bob of mass m_2 suspended by a massless rod of length l_2 from an 'upper' bob of mass m_1 which is in turn suspended from a fixed pivot by a massless rod of length l_1 (see figure 1). The system has 2 degrees of freedom, it is free to move in a vertical plane subject to gravity. The rods make angles θ_1, θ_2 counterclockwise relative to the downward vertical. The cartesian coordinates of the two bobs are

$$\begin{aligned} \mathbf{r}_1 &= (x_1, y_1) \quad \text{where} \quad x_1 = l_1 \sin\theta_1 \quad \text{and} \quad y_1 = -l_1 \cos\theta_1, \quad \text{and} \\ \mathbf{r}_2 &= (x_2, y_2) \quad \text{where} \quad x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad \text{and} \quad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2. \end{aligned} \tag{67}$$

• Assuming the potential energy vanishes at the height of the pivot, the potential and kinetic energies are

$$\begin{aligned} V &= -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \quad \text{and} \\ T &= \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2}\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2 l_1 l_2 c_{12} \dot{\theta}_1 \dot{\theta}_2\right]. \end{aligned} \tag{68}$$

Here we abbreviate $s_{12} = \sin(\theta_1 - \theta_2)$ and $c_{12} = \cos(\theta_1 - \theta_2)$. To simplify things, we take bobs of equal masses m and rods of equal length l. In this case, $|V| \leq 3mgl$ while $0 \leq T < \infty$.

• The configuration space of the double pendulum is a torus $\mathbb{T}^2 = S^1 \times S^1$ with coordinates $\theta_1 \in S^1, \theta_2 \in S^1$. The Lagrangian is

$$L = T - V = \frac{1}{2} m l^2 \left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 c_{12} \dot{\theta}_1 \dot{\theta}_2\right] + mgl\left[2\cos\theta_1 + \cos\theta_2\right]. \tag{69}$$

The momenta conjugate to θ_1, θ_2 are

$$p_1 = \frac{\partial L}{\partial \theta_1} = ml^2 \left[2\dot\theta_1 + c_{12}\dot\theta_2\right] \quad \text{and} \quad p_2 = \frac{\partial L}{\partial \theta_2} = ml^2 \left[\dot\theta_2 + c_{12}\dot\theta_1\right]. \tag{70}$$

The conjugate momenta do not coincide with the angular momenta of the two bobs, though their sum coincides with the total angular momentum of the pendulum. The angular momenta are

$$\mathbf{L}_1 = m\mathbf{r}_1 \times \dot{\mathbf{r}}_1 = ml^2\dot\theta_1\, \hat z \quad \text{and} \quad \mathbf{L}_2 = m\mathbf{r}_2 \times \dot{\mathbf{r}}_2 = ml^2 \left[\dot\theta_1 + \dot\theta_2 + c_{12}\left(\dot\theta_1 + \dot\theta_2\right)\right] \hat z \tag{71}$$

$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = (p_1 + p_2)\hat z$. We will use the conjugate momenta p_1, p_2 rather than the angular momenta.

• The 'generalized forces' are

$$\frac{\partial L}{\partial \theta_1} = -ml\left[2g\sin\theta_1 + ls_{12}\,\dot\theta_1\dot\theta_2\right] \quad \text{and} \quad \frac{\partial L}{\partial \theta_2} = ml\left[ls_{12}\,\dot\theta_1\dot\theta_2 - g\sin\theta_2\right]. \tag{72}$$

Lagrange's equations of motion are a pair of second order non-linear ODEs

$$2\ddot\theta_1 + c_{12}\,\ddot\theta_2 + s_{12}\,\dot\theta_2^2 + 2\omega^2\sin\theta_1 = 0 \qquad \text{and} \qquad \ddot\theta_2 + c_{12}\,\ddot\theta_1 - s_{12}\,\dot\theta_1^2 + \omega^2\sin\theta_2 = 0. \tag{73}$$

They involve only one material parameter $\omega^2 = g/l$. Upon expressing the generalized coordinates in terms of momenta,

$$\dot\theta_1 = \frac{p_1 - c_{12}p_2}{ml^2(1+s_{12}^2)} \quad \text{and} \quad \dot\theta_2 = \frac{2p_2 - c_{12}p_1}{ml^2(1+s_{12}^2)} \tag{74}$$

we find the conserved hamiltonian $H = p_1\dot\theta_1 + p_2\dot\theta_2 - L = T + V$

$$H = \frac{1}{2ml^2(1+s_{12}^2)}\left[p_1^2 + 2p_2^2 - 2c_{12}p_1p_2\right] - mgl[2\cos\theta_1 + \cos\theta_2]. \tag{75}$$

The conserved energy may also be expressed in terms of coordinates and velocities:

$$E = \frac{1}{2}ml^2\left[2\dot\theta_1^2 + \dot\theta_2^2 + 2c_{12}\dot\theta_1\dot\theta_2\right] - mgl[2\cos\theta_1 + \cos\theta_2]. \tag{76}$$

The phase space of the double pendulum is four dimensional, with coordinates $\theta_1 \in S^1, \theta_2 \in S^1, p_1, p_2 \in \mathbb{R}$. The phase space is the cartesian product of a torus and a plane $\mathbb{T}^2 \times \mathbb{R}^2$

• Besides energy, the double pendulum does not possess any obvious conserved quantity. However, when the energy is very large, most of it is kinetic since the gravitational potential energy is bounded between $\pm 3mgl$. For example, the two bobs could just go round very fast in uniform circular motion. So in the limit of high energies ($E \gg 3mgl$) we should be able to ignore the gravitational force, and the torque it imparts. As a consequence, total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ should be conserved at asymptotically high energies. We already know that

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = ml^2\left[2\dot\theta_1 + \dot\theta_2 + 2c_{12}(\dot\theta_1 + \dot\theta_2)\right]\hat z = (p_1 + p_2)\hat z \tag{77}$$

This expression for the conserved total angular momentum may also be obtained using Noether's theorem. The Lagrangian ignoring gravity

$$L = T = \frac{1}{2}ml^2\left[2\dot\theta_1^2 + \dot\theta_2^2 + 2\cos(\theta_1 - \theta_2)\dot\theta_1\dot\theta_2\right] \tag{78}$$

is invariant under (infinitesimal) rotations $\theta_1 \to \theta_1 + \delta\phi, \theta_2 \to \theta_2 + \delta\phi$. Noether's theorem guarantees conservation of

$$p_1\,\delta\theta_1 + p_2\,\delta\theta_2 = \delta\phi\, ml^2[2\dot{\theta}_1 + c_{12}\dot{\theta}_2] + \delta\phi\, ml^2[\dot{\theta}_2 + c_{12}\dot{\theta}_1] \tag{79}$$

Since $\delta\phi$ is arbitrary, we may omit it and get an expression for the conserved angular momentum

$$\mathbf{L} = (p_1 + p_2)\hat{z} = ml^2\left[(2 + c_{12})\dot{\theta}_1 + (1 + c_{12})\dot{\theta}_2\right]\hat{z} \tag{80}$$

Numerical solutions of the equations of motion of the double pendulum show that $\mathbf{L}$ fluctuates around a mean value. As the energy increases, the fluctuations in $\mathbf{L}$ get smaller, and in the limit of infinite energy, angular momentum is exactly conserved just as for the *simple pendulum* (see fig.2).

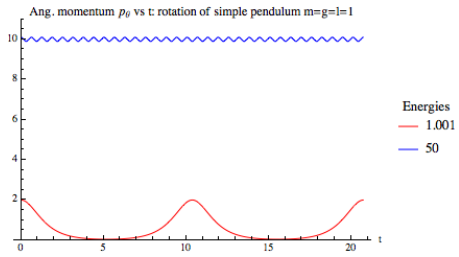


Figure 2: Angular momentum vs time for simple pendulum rotational motion () $m = l = g = 1$ and low and high energies.

5.2 Small oscillations of a double pendulum: normal modes

• In general, it has not been possible to solve the equations of motion of a double pendulum in closed form due to their non-linearities (not even with elliptic functions! The motion is chaotic!). However, if the deflection angles are always small, we may linearize the equations of motion and solve them. The motion reduces to the integrable dynamics of a pair of coupled harmonic oscillators. Let us see why.

• If both $|\theta_1|, |\theta_2| \ll 1$ we may approximate the trigonometric functions cos and sin by their quadratic Taylor polynomials in the kinetic and potential energies, so that the resulting equations of motion become linear. The Lagrangian becomes

$$L = \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right] + mgl\left[3 - \theta_1^2 - \frac{1}{2}\theta_2^2\right] = T - V. \tag{81}$$

We omit the constant $3mgl$ from the Lagrangian: it doesn't affect the eom. The conjugate momenta are

$$p_1 = ml^2\left(2\dot{\theta}_1 + \dot{\theta}_2\right) \quad \text{and} \quad p_2 = ml^2\left(\dot{\theta}_1 + \dot{\theta}_2\right). \tag{82}$$

and

$$\dot{\theta}_1 = \frac{p_1 - p_2}{ml^2} \quad \text{and} \quad \dot{\theta}_2 = \frac{2p_2 - p_1}{ml^2}. \tag{83}$$

The equations of motion depend only on one physical parameter $\omega^2 = g/l$:

$$2\ddot{\theta}_1 + \ddot{\theta}_2 + 2\omega^2\theta_1 = 0 \quad \text{and} \quad \ddot{\theta}_1 + \ddot{\theta}_2 + \omega^2\theta_2 = 0. \tag{84}$$

The corresponding conserved energy is $H = T + V$,

$$H = \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right] + mgl\left[\theta_1^2 + \frac{\theta_2^2}{2}\right] \;=\; \frac{1}{2ml^2}\left[p_1^2 + 2p_2^2 - 2p_1p_2\right] + mgl\left[\theta_1^2 + \frac{\theta_2^2}{2}\right]. \quad (85)$$

The equations of motion are now a pair of coupled second order linear ODEs with constant coefficients. It is possible to change variables to normal modes to get a pair of de-coupled linear ODEs. Let us first write the eom in matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix}2 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = -2\omega^2\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \quad (86)$$

If we let B be the constant coefficient matrix,

$$B = \begin{pmatrix}2 & 1\\ 2 & 2\end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \quad \text{then} \quad \frac{d^2}{dt^2}B\theta = -2\omega^2\theta \quad (87)$$

Though B is not a symmetric matrix, it has distinct eigenvalues $\lambda_\pm = 2 \pm \sqrt{2}$, and therefore can be diagonalized. The corresponding eigenvectors are not orthogonal, but may be taken as

$$a_+ = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix} \quad \text{and} \quad a_- = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}. \quad (88)$$

B may be diagonalized by a (non-orthogonal) similarity transformation S whose matrix representation has columns that are the eigenvectors of B

$$S^{-1}BS = D \quad \text{where} \quad S = \frac{1}{2}\begin{pmatrix}1 & 1\\ \sqrt{2} & -\sqrt{2}\end{pmatrix}, \; S^{-1} = \begin{pmatrix}1 & \frac{1}{\sqrt{2}}\\ 1 & -\frac{1}{\sqrt{2}}\end{pmatrix} \text{ and } D = \begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}. \quad (89)$$

The equations of motion become

$$\frac{d^2}{dt^2}SDS^{-1}\theta = -2\omega^2\theta \;\; \Rightarrow \;\; \frac{d^2}{dt^2}(S^{-1}\theta) = -2\omega^2 D^{-1}(S^{-1}\theta) \quad (90)$$

If we denote

$$S^{-1}\theta = \xi = \begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}\theta_1 + \frac{\theta_2}{\sqrt{2}}\\ \theta_1 - \frac{\theta_2}{\sqrt{2}}\end{pmatrix} \quad \text{and} \quad 2\omega^2 D^{-1} = \begin{pmatrix}\frac{2\omega^2}{\lambda_+} & \\ 0 & \frac{2\omega^2}{\lambda_-}\end{pmatrix} = \begin{pmatrix}\omega_+^2 & 0\\ 0 & \omega_-^2\end{pmatrix}, \quad (91)$$

then the components $\xi_\pm$ evolve via *decoupled* 2nd order ODEs

$$\ddot{\xi}_+(t) = -\omega_+^2\xi_+(t) \quad \text{and} \quad \ddot{\xi}_-(t) = -\omega_-^2\xi_-(t) \quad \text{where} \quad \omega_\pm^2 = \frac{2\omega^2}{2 \pm \sqrt{2}}. \quad (92)$$

$\xi_\pm(t)$ are called normal modes of the system, they are periodic functions of time with periods

$$T_\pm = \frac{2\pi}{\omega_\pm} = \frac{2\pi}{\omega}\sqrt{1 \pm \frac{1}{\sqrt{2}}}. \quad (93)$$

$\xi_\pm(t)$ may be expressed in terms of trigonometric functions of time

$$\xi_+(t) = c_1\cos(\omega_+ t) + c_2\sin(\omega_+ t) \quad \text{and} \quad \xi_-(t) = c_3\cos(\omega_- t) + c_4\sin(\omega_- t). \quad (94)$$

The four coefficients c_i are to be fixed using the initial conditions. The original deflection angles are determined via $\theta = S\xi$

$$\theta_1 = \frac{1}{2}(\xi_+ + \xi_-) \quad \text{and} \quad \theta_2 = \frac{1}{\sqrt{2}}(\xi_+ - \xi_-). \tag{95}$$

Note that the general motion of the double pendulum in the small angle approximation is not periodic. The above solution is a linear combination of periodic functions whose periods are not in rational ratio

$$\frac{T_+}{T_-} = \frac{\omega_-}{\omega_+} = \sqrt{\frac{\lambda_+}{\lambda_-}} = 1 + \sqrt{2} \notin \mathbb{Q} \tag{96}$$

In general, the motion is quasi-periodic. The double pendulum does not return to its initial state, but approaches it arbitrarily closely if we are willing to wait long enough. However, if initial conditions are chosen so that only one of the two normal modes ξ_+ or ξ_- is present (e.g. if $c_3 = c_4 = 0$), then the motion is periodic.

• We may use the normal modes to find a new conserved quantity for small oscillations of a double pendulum. As for the simple pendulum or harmonic oscillator, from the equations of motion,

$$\ddot{\xi}_+ = -\omega_+^2 \xi_+ \quad \text{and} \quad \ddot{\xi}_- = -\omega_-^2 \xi_- \tag{97}$$

we infer that the energy of each normal mode is a constant of motion

$$H_+ = \frac{1}{2}ml^2\left[\dot{\xi}_+^2 + \omega_+^2\xi_+^2\right] \quad \text{and} \quad H_- = \frac{1}{2}ml^2\left[\dot{\xi}_-^2 + \omega_-^2\xi_-^2\right]. \tag{98}$$

The pre-factor ml^2 is chosen so that $H_\pm$ have dimensions of energy. The total energy

$$H = \frac{1}{2}ml^2[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2] + mgl[\theta_1^2 + \frac{1}{2}\theta_2^2] \tag{99}$$

is of course also conserved. Are $H, H_\pm$ functionally independent? This is unlikely since we would expect the total energy to be a sum of energies contributed by the various normal modes, which do not interact with each other. In fact, we will show that H is a weighted sum of the energies of the normal modes $H = 2\lambda_+ H_+ + 2\lambda_- H_-$. To see this we write the total energy as a quadratic form and express θ_i in terms of normal modes ξ_i:

$$\begin{aligned} H &= \frac{1}{2}ml^2 \begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}^t \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} + mgl \begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}^t \begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} = \frac{1}{2}ml^2\dot{\theta}^t\tau\dot{\theta} + mgl\theta^t v\theta \\ \text{where} \qquad \tau &= \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}. \end{aligned} \tag{100}$$

The kinetic and potential matrices τ and v are not uniquely defined. But if they are chosen symmetric, then they are unique. We may add any anti-symmetric matrices to τ and v without affecting the formula for energy. Writing $\theta = S\xi$ and using $S^t\tau S = 2D = 2\text{diag}(\lambda_+, \lambda_-)$ and $S^t v S = 2I$ we get

$$\begin{aligned} H &= \frac{1}{2}ml^2\, \dot{\xi}^t\, (S^t\tau S)\, \dot{\xi} + mgl\, \xi^t\, (S^t v S)\, \xi \\ &= (2\lambda_+)\frac{1}{2}ml^2\dot{\xi}_+^2 + (2\mu_-)\frac{1}{2}ml^2\dot{\xi}_-^2 + (2\lambda_+)\frac{mgl}{\lambda_+}\xi_+^2 + (2\lambda_-)\frac{mgl}{\lambda_-}\xi_-^2 \end{aligned}$$

$$= \quad 2\lambda_+ H_+ + 2\lambda_- H_-. \tag{101}$$

In the last step we used $\omega_\pm^2 = 2\omega^2/\lambda_\pm$ to write

$$\frac{mgl}{\lambda_\pm} = \frac{ml^2\omega^2}{\lambda_\pm} = \frac{1}{2} ml^2\, \omega_\pm^2 \tag{102}$$

So of H, H_1 and H_2, only two are independent conserved quantities. Thus we have identified a second conserved quantity for small oscillations of a double pendulum. In the limit of low energies, we find that the motion of a double pendulum is integrable, we have explicitly found the general solution.

6 Vibrations of a stretched string

6.1 Wave equation for transverse vibrations of a stretched string

• Perhaps the simplest physically interesting mechanical system with a continuously infinite number of degrees of freedom is a vibrating stretched string. We will consider the special case where, in equilibrium, the string is stretched between two clamps located at $x = 0$ and $x = L$. We ignore the effects of gravity since the tensional forces in the string often dominate. We shall call the direction in which the string is stretched, the 'horizontal' direction. The string is free to move only in one direction (vertical) transverse to the direction in which the string is stretched. We assume the string has a mass per unit length of ρ. The instantaneous configuration of the string is specified by giving the height $u(x,t)$ of the string above the horizontal position x at time t. Since the string is stretched, there are tension forces that act on any segment of the string, tangentially at either end of the segment, tending to elongate the segment. It is usually assumed that the tension in the string is a constant τ, though we will allow it to vary slowly with location, so $\tau = \tau(x)$. When the string is horizontal, the tensions at either end of any segment are horizontal, equal and opposite in direction so that the string is in equilibrium. At the end points, the tension is balanced by the force applied by the clamps.

• Note that the length of the string is not fixed, it can stretch to a length more than L, for instance when it is plucked as in a Veena. When the string is displaced from equilibrium by small vertical displacements, tensional forces on the ends of a small segment are not necessarily horizontal. But to a good approximation, the horizontal components of tension are equal and opposite, ensuring that there is no longitudinal/horizontal movement of the string. Moreover, the vertical components of tension are in general unequal and result in a vertical acceleration of the segment. We estimate this. Consider a small segment of string between horizontal locations x and $x + dx$ with corresponding heights $u(x)$ and $u(x + dx) \approx u + du$. We suppose that the tangent to the string at any point x makes a counter-clockwise angle $\theta(x)$ with respect to the horizontal. Draw a diagram! Then since we assume the inclination angles are small,

$$\cos\theta(x) \approx 1 - \frac{\theta(x)^2}{2} \approx 1 \quad \text{and} \quad \sin\theta(x) \approx \tan\theta(x) \approx \frac{\partial u}{\partial x} \equiv u'(x). \tag{103}$$

Then the horizontal components of tension at the right and left ends of the segment are $\tau(x + dx)\cos\theta(x + dx)$ and $-\tau(x)\cos\theta(x)$. Since we are assuming that the string does not move horizontally, these must be equal and opposite (If we approximate $cos\theta \approx 1$, then this is possible only if τ is independent of x).

The vertical components of tension at the right and left ends of the segment are

$$\tau(x+dx)\sin\theta(x+dx)\hat{z} \approx (\tau u')(x+dx)\hat{z} \quad \text{and} \quad -\tau(x)\sin\theta(x)\hat{z} \approx -(\tau u')(x)\hat{z}. \tag{104}$$

Thus the net upward force on the segment is

$$F_{\text{up}} = (\tau u')(x+dx) - (\tau u')(x) \approx \frac{\partial(\tau(x)u'(x))}{\partial x}\, dx \tag{105}$$

So Newton's second law for the segment, whose mass is $\rho(x)\ dx$ is (subscripts denote partial derivatives)

$$F_{\text{up}} = (\tau u_x)_x\, dx = \rho\, dx\, u_{tt}. \tag{106}$$

Thus the equation of motion for small transverse (1D) vibrations of the stretched string is $(\tau u_x)_x = \rho u_{tt}$. If the tension τ is a constant, then we get the standard form of the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{where} \quad c = \sqrt{\frac{\tau}{\rho}} = \sqrt{\frac{\text{tension}}{\text{mass per unit length}}}. \tag{107}$$

c has dimensions of speed and will be seen to be the speed at which waves propagate on the string. The wave equation $u_{tt} = c^2 u_{xx}$ is a linear, homogeneous partial differential equation second order in both space and time derivatives. PDEs involve derivatives with respect to more than one independent variable (x, t here) while ordinary differential equations (ODEs) involve derivatives only in on independent variable (t in Newton's equation for a point particle). The wave equation is linear since it involves only the first power of the unknown function (dependent quantity) u.

• We are interested in solving the initial-boundary value problem for the string. The wave equation is second order in time and requires two initial conditions (say at $t = 0$), just like Newton's equation. These are the initial height $u(x, t = 0)$ and the initial velocity of the string $\dot{u}(x, t = 0)$. In addition, we need to specify what happens at the boundaries. The boundary conditions corresponding to a string clamped at the end points are $u(x = 0, t) = u(x = L, t) = 0$. This is called Dirichlet boundary conditions. Other boundary conditions are also of interest. For example, we might have an end (say at $x = 0$) of the string free to move up and down (though not horizontally), so that the slope of the string vanishes at the end point. This could be implemented by attaching the left end of the string to a massless ring free to move vertically with out friction on a pole. This means u has no slope at the left end point, one cannot apply a vertical force on the ring since it yields, it has no inertia. This leads to the free/open/Neumann boundary condition $\frac{\partial u}{\partial x} = 0$ at $x = 0$. We could also consider an infinite string with say, $u(x, t) \to 0$ as $|x| \to \infty$.

• As a consequence of considering small vibrations and small angles θ the equation of motion is linear, however, it is a partial differential equation unlike Newton's ordinary differential equations encountered in the mechanics of finitely many particles. Above, $u(x)$ is the analogue of the generalised coordinate and x labels the particles in the string. The configuration space is the set of possible instantaneous locations of the string segments, i.e. the space of twice differentiable functions $u(x)$ on the interval $[0, L]$ that vanish at the end-points. This is an infinite dimensional space reflecting the fact that a string has infinitely many degrees of freedom. The equations of continuum mechanics (e.g. fluid mechanics, electrodynamics, general relativity, elasticity) are typically systems of partial differential equations and the wave equation is perhaps the simplest prototype. We may regard a partial differential equation such as the wave equation as a large (infinite) system of ODEs, one ODE for each value of x.

6.2 Separation of variables and normal modes of a vibrating string

• On account of the linearity of the wave equation, the superposition principle applies. Linear combinations of solutions are again solutions. The solution space forms a linear vector space. This suggests that if we can find a sufficiently large set of linearly independent solutions (called normal modes of oscillation), we may be able to express a solution of interest as a linear combination of the normal modes of oscillation.

• The wave equation $u_{tt} = c^2 u_{xx}$ is a partial differential equation for an unknown height function u dependent on two independent variables t, x. Let us look for solutions which are a product of a function of t alone and a function of x alone: $u(x,t) = X(x)T(t)$. We hope that solutions of this separable type form a basis for the space of all solutions of interest. We also hope that X and T will be determined by simpler ODEs compared to the PDE for u. Indeed, we find, wherever the quotients make sense,

$$X(x)\ddot{T}(t) = c^2 T(t)X''(x) \quad \Rightarrow \quad \frac{\ddot{T}(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = -\omega^2. \tag{108}$$

Now LHS is a function of t alone while RHS is a function of x alone. Thus, both must equal the same constant which we called $-\omega^2$. We anticipate that the constant must be negative for physically interesting vibrational motion. This is because $-\omega^2 = \frac{\ddot{u}}{u}$ is the ratio of the acceleration of the string element to its displacement from the mean position. As in Hooke's law, this quotient must be negative for a restoring force. Thus, our PDE has reduced to a pair of ODEs

$$\ddot{T}(t) = -\omega^2 T(t) \quad \text{and} \quad X''(x) = -k^2 X(x) \quad \text{where} \quad k = \frac{\omega}{c} \text{ is the angular wave number.} \tag{109}$$

These ODEs are in fact eigenvalue problems. The first is the same as Newton's equation for the harmonic oscillator and the second is essentially the same, so we can write their general solutions as

$$T(t) = A\cos\omega t + B\sin\omega t \quad \text{and} \quad X(x) = C\cos kx + D\sin kx. \tag{110}$$

The clamping of end points of the string (Dirichlet boundary conditions) implies $X(0) = X(L) = 0$ so we must have $C = 0$ and $\sin kL = 0$. So $kL = n\pi$ where n is an integer, it suffices to take $n \geq 1$ since the negative values give (linearly dependent) solutions that only differ by a sign and $n = 0$ gives the trivial solution. So we may write any separable solution of the wave equation as

$$u_n(x,t) = (A\cos\omega_n t + B\sin\omega_n t)\sin\frac{n\pi x}{L} \quad \text{where} \quad \omega_n = \frac{n\pi c}{L} \quad \text{for some} \quad n = 1, 2, 3, \ldots \tag{111}$$

Each of these solutions for $n = 1, 2, 3, \ldots$ is called a normal mode of oscillation. The mode $n = 1$ is called the fundamental or first harmonic $n = 2$ the second harmonic or first overtone etc. A normal mode of oscillation has a definite angular wave number k_n and spatial wave length $\lambda_n = 2\pi/k_n = 2L/n$. It has a definite angular frequency $\omega_n = ck_n$ and also a definite time period of oscillation $T_n = 2\pi/\omega_n$. $\nu_n = \omega/2\pi$ is the frequency at which every point along the string vibrates about its mean position. As opposed to a normal mode, a more general motion of a stretched string will not have such a definite wave length and time period, indeed it need not even be periodic in time! Moreover, these normal modes do not necessarily satisfy the prescribed

initial conditions. But since the wave equation is linear, we can take linear combinations of normal modes to produce new solutions. The most general such linear combination is a Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \sin \frac{n\pi x}{L} \tag{112}$$

The coefficients A_n, B_n must decay sufficiently fast as $n \to \infty$ to ensure that the sum converges. The theorems of Fourier series tell us that we can represent any continuous function that vanishes at the end points of the interval $[0, L]$ as a Fourier sine series. So we may try to fit the initial conditions by a suitable choice of constants A_n, B_n for $1 \leq n \leq \infty$. They are fixed by the initial height and velocity of the string

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \quad \text{and} \quad \dot{u}(x,0) = \sum_{n=1}^{\infty} B_n \omega_n \sin \frac{n\pi x}{L} \quad \text{where} \quad \omega_n = \frac{n\pi c}{L}. \tag{113}$$

Using the orthogonality of $\sin(n\pi x/L)$ on the interval $[0, L]$ for $n = 1, 2, 3, \ldots$ and the fact that the average value of the square of the sine function is a half, we find

$$A_n = \frac{2}{L} \int_0^L u(x,0) \sin\left(\frac{n\pi x}{L}\right)\, dx \quad \text{and} \quad B_n = \frac{2}{n\pi c} \int_0^L \dot{u}(x,0) \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{114}$$

Thus we have solved the initial-boundary value problem for the motion of a stretched string clamped at the end points. It is instructive to plot a movie of the time evolution of one such solution on a computer.

• We see that a general vibration of a stretched string involves a superposition of several normal modes and does not possess a definite wave number or time period. However, we will see that in general, higher harmonics cost more energy to excite. We might anticipate this since the restoring force was found to be proportional $u''(x)$. Higher harmonics $\sin(n\pi x/L)$, for $n \gg 1$ are rapidly oscillating functions with large second derivatives, so they involve significant forces on the string segments. We would expect much energy to be stored in the oscillatory motion of a higher harmonic.

6.3 Conserved energy of small oscillations of a stretched string

• Since we have not incorporated any dissipative effects and are not supplying any energy to the string at any time $t > 0$, we expect the energy of the vibrating string to the conserved. Let us derive an expression for the conserved energy in the same way as we did for Newton's equation. Recall that we multiplied $m\ddot{q}_i + \frac{\partial V}{\partial q_i} = 0$ by the integrating factor $\dot{q}_i$ and summed over the degrees of freedom i. The resulting expression was the statement that the time derivative of energy is zero.

• So let us begin with Newton's equation for a string in its pristine form and multiply by u_t

$$\rho u_{tt}\, dx = (\tau u_x)_x\, dx \quad \Rightarrow \quad \rho u_t u_{tt}\, dx - u_t(\tau u_x)_x\, dx = 0 \quad \Rightarrow \quad \frac{1}{2}\rho(u_t^2)_t\, dx - u_t(\tau u_x)_x\, dx = 0. \tag{115}$$

Now we sum over the degrees of freedom by integrating over $x \in [a, b]$

$$\partial_t \int_a^b \frac{1}{2}\rho u_t^2\, dx - \int_a^b u_t(\tau u_x)_x = 0. \tag{116}$$

The first term is the time derivative of what looks like a kinetic energy by analogy with a point particle

$$\frac{m}{2}\sum_i \dot{q}_i^2 \to \int_a^b \frac{1}{2}\,\rho\, u_t^2. \tag{117}$$

So we would like to express the second term as the time derivative of a potential energy. To do so we first integrate by parts

$$\partial_t \int_a^b \frac{1}{2}\rho u_t^2\; dx - [\tau u_t u_x]_a^b + \int_a^b \tau u_x u_{tx}\; dx = 0 \tag{118}$$

The boundary term vanishes if we use Dirichlet or free boundary conditions ($u = 0$ or $u_x = 0$ at $x = a, b$) or even periodic boundary conditions ($\tau(a) = \tau(b)$, $u(a,t) = u(b,t)$, $u_x(a,t) = u_x(b,t)$) and we get

$$\partial_t \int_a^b \rho\frac{u_t^2}{2}\; dx + \partial_t \int_a^b \frac{1}{2}\tau(u_x^2)\; dx = 0. \tag{119}$$

Thus the conserved energy is a sum of kinetic and potential energies (check the dimensions!)

$$E = T + V = \int_a^b \left[\frac{1}{2}\rho u_t^2 + \frac{1}{2}\tau u_x^2\right]\; dx = \int_a^b \mathcal{E}\; dx \quad \text{with} \quad \frac{dE}{dt} = 0. \tag{120}$$

The KE is proportional to the sum of squares of speeds of the bits of string as expected. The PE is an energy of 'bending', it is proportional to the square of the gradient (slope) of the string profile. T and V are separately non-negative and so $E \geq 0$ with equality when the stretched string is in equilibrium $u(x,t) =$ constant. The integrand is called the energy density $E = \int \mathcal{E}(x,t)dx$. In general, the energy density 'moves around the string' in such a way that the total energy is conserved. We also see that for fixed A, B, higher ($n \gg 1$) normal modes of oscillation $u_n = [A\sin(n\pi ct/L) + B\cos(n\pi ct/L)]\sin(n\pi x/L)$ store more energy.

• We verify that the energy is conserved using the eom $\rho u_{tt} = (\tau u_x)_x$ and integration by parts

$$\frac{dE}{dt} = \int [\rho u_t u_{tt} + \tau u_x u_{xt}]\; dx = \int \left[\rho u_t \frac{1}{\rho}(\tau u_x)_x - (\tau u_x)_x u_t + \partial_x(\tau u_x u_t)\right]\; dx = [\tau u_x u_t]_0^L = 0. \tag{121}$$

We assumed the boundary term $[\tau u_t u_x]_0^L$ vanishes. This is automatic if u or u_x vanish at the end points, which is the case for a clamped string (Dirichlet b.c.) or a string with free boundary conditions ($u_x = 0$). Thus the energy of the string is conserved. The energy was initially supplied to the string when it was set in motion through the initial gradients in the string profile $u_x(t = 0)$ and initial velocity of the string $u_t(t = 0)$.

6.4 Lagrangian and Hamiltonian for stretched string

• The possible instantaneous configurations of a vibrating stretched string are the heights $u(x)$ for $0 \leq x \leq L$. So the configuration space is a space of functions, it is not finite dimensional. The generalised coordinates are the values of the function $u(x,t)$ for $0 \leq x \leq L$ at a given time t. The generalised velocities at time t are $\dot{u}(x,t)$. We will show that a Lagrangian for small transverse oscillations of the stretched string is

$$L = \int_0^L \frac{1}{2}\left[\rho(u_t)^2 - \tau(u_x)^2\right]\; dx \equiv \int \mathcal{L}\; dx \tag{122}$$

where $\mathcal{L} = \frac{1}{2}\left[\rho u_t^2 - \tau u_x^2\right]$ is called the Lagrangian density. This formula is simply $L = T - V$ obtained from our earlier formula for the conserved energy $E = T + V$.

• In general, for a Lagrangian density $\mathcal{L}$ that depends on u and its time and space derivatives $u_t,\ u_x, u_{xx}$, the Euler-Lagrange equation of motion (assuming suitable boundary conditions) is

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right). \tag{123}$$

To see this, we simply set the first variation of the action $S = \int \mathcal{L} dx\, dt$ to zero after integrating by parts

$$\begin{aligned}
\delta S &= \int \left[\frac{\partial \mathcal{L}}{\partial u}\delta u + \frac{\partial \mathcal{L}}{\partial u_t}\delta u_t + \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x + \frac{\partial \mathcal{L}}{\partial u_{xx}}\delta u_{xx}\right] dx\, dt \\
&= \int \left[\frac{\partial \mathcal{L}}{\partial u} - \partial_t\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - \partial_x\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right] \delta u\, dx\, dt \\
\delta S = 0 \ \Rightarrow\ & \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right).
\end{aligned} \tag{124}$$

For the above Lagrangian density we get $\partial_t(\rho u_t) = \partial_x(\tau u_x)$ or $\rho u_{tt} = \partial_x(\tau u_x)$ since ρ is not explicitly time dependent. When ρ, τ are constants, this reduces to the familiar form of the wave equation $u_{tt} = c^2 u_{xx}$ with $c^2 = \tau/\rho$.

• The momentum conjugate to the coordinate $u(x,t)$ is

$$\pi(x,t) = \frac{\delta L}{\delta u_t(x)} = \rho u_t(x,t). \tag{125}$$

Thus the hamiltonian is

$$H[u,\pi] = \mathrm{ext}_{u_t}\int_0^L \left[\pi u_t - \mathcal{L}\right]\, dx = \int \left[\frac{\pi^2}{\rho} - \frac{1}{2}\rho\frac{\pi^2}{\rho^2} + \frac{1}{2}\tau u_x^2\right]\, dx = \int_0^L \left[\frac{\pi(x)^2}{2\rho} + \frac{\tau u_x^2}{2}\right]\, dx. \tag{126}$$

And the Poisson brackets between canonically conjugate variables (at a common time t) are

$$\{u(x), \pi(x')\} = \delta(x - x') \quad \text{and} \quad \{u(x), u(x')\} = \{\pi(x), \pi(x')\} = 0. \tag{127}$$

• Let us obtain Hamilton's equations and check that they reduce to the wave equation. Integrating by parts assuming clamped or free boundaries, we may express the hamiltonian as

$$H[u,\pi] = \int \left[\frac{1}{2\rho}\pi^2 - \frac{1}{2}u(\tau u_x)_x\right]\, dx \quad \text{assuming} \quad [\tau u u_x]_0^L = 0. \tag{128}$$

Hamilton's equations are

$$u_t(x) = \frac{\delta H}{\delta \pi(x)} = \frac{\pi(x)}{\rho} \quad \text{and} \quad \pi_t(x) = -\frac{\delta H}{\delta u(x)} = (\tau u_x)_x \tag{129}$$

Combining these two 1st order equations, we get the 2nd order wave equation $\rho u_{tt} = (\tau u_x)_x$ or $u_{tt} = c^2 u_{xx}$ for constant tension, as expected.

6.5 Right- and left-moving waves and d'Alembert's solution

• By playing with a stretched string, we may discover the phenomenon of a transverse wave that moves along a string, a traveling wave. A vertical disturbance that is set up somewhere along a string can propagate elsewhere. This is because the vertical component of tension causes the neighboring string element to move vertically, and the process goes on. Indeed, such a traveling wave can reach a boundary (clamp) and get reflected and come back. Two such traveling waves moving in opposite directions can collide and superpose. This suggests that it may be possible to describe the solution of the wave equation in terms of traveling waves.

• The height $u(x,t)$ (measured relative to the equilibrium height) of a stretched string executing small transverse vibrations must satisfy the wave equation $\Box u = (\frac{1}{c^2}\partial_t^2 - \partial_x^2)u = 0$. In other words, it must be annihilated by the wave operator or d'Alembertian $\Box$. d'Alembert's approach to solving the wave equation arises from factorizing the wave operator $\Box$ into a pair of first order operators. Let us consider the wave equation on an infinite interval $-\infty < x < \infty$ subject to the initial height and initial velocity

$$u(x, t = 0) = h(x) \quad \text{and} \quad \dot{u}(x, 0) = v(x). \tag{130}$$

The wave equation may be factorized as

$$\left(c^{-2}\partial_t^2 - \partial_x^2\right) u = \left(c^{-1}\partial_t - \partial_x\right)\left(c^{-1}\partial_t + \partial_x\right) u = \left(c^{-1}\partial_t + \partial_x\right)\left(c^{-1}\partial_t - \partial_x\right) u = 0 \tag{131}$$

It follows that if u is annihilated by either $\partial_- = c^{-1}\partial_t - \partial_x$ or $\partial_+ = c^{-1}\partial_t + \partial_x$, then it will satisfy the wave equation[4]. Let us consider these first order equations. We notice that any differentiable function $u(x,t) = f(x - ct)$ satisfies $(c^{-1}\partial_t + \partial_x)u = 0$ while any differentiable function $u(x,t) = g(x + ct)$ is annihilated by $c^{-1}\partial_t - \partial_x$. Thus, for any differentiable functions f and g,

$$u(x,t) = f(x - ct) + g(x + ct) \tag{132}$$

is a solution of the wave equation. A little thought shows that for $c > 0$, $f(x - ct)$ is a right-moving wave with speed c and initial profile (at $t = 0$) given by the function $f(x)$. The shape of the wave $f(x-ct)$ is unaltered as it travels to the right. So $f(x-ct)$ is called a right-moving wave. Similarly, for $c > 0$, $g(x + ct)$ is a left-moving wave. Thus we have found that any superposition of a right- and left-moving wave is a solution of the wave equation.

• One wonders whether such superpositions of right and left moving waves are adequate to solve the initial value problem for a stretched string[5]. We will see that this is indeed the case on an infinite domain. To solve the IVP, we wish to fix f and g in terms of the initial data.

$$u(x,0) = f(x)+g(x) = h(x) \;\text{ and }\; \dot{u}(x,0) = -cf'(x)+cg'(x) = v(x) \;\text{ or }\; -f'(x)+g'(x) = \frac{1}{c}v(x).$$

Integrating the latter equation with integration constant K we get

$$f(x) + g(x) = h(x) \quad \text{and} \quad -f(x) + g(x) = \frac{1}{c}\int_{x_0}^{x} v(\xi)\, d\xi + K. \tag{133}$$

[4]These are not necessary conditions for solving the wave equation, only sufficient. But functions of these special sorts can be used to obtain the complete solution to the initial value problem as we will see soon.

[5]It can be shown (try!) that the initial value problem for the wave equation has a unique solution. So the solution we find here in terms of left- and right-moving waves and expressed in terms of initial height and initial velocity is the only one.

Adding and subtracting we solve for f, g in terms of initial data

$$f(x) = \frac{1}{2}\left(h(x) - \frac{1}{c}\int_{x_0}^{x} v(\xi)\, d\xi - K\right) \quad \text{and} \quad g(x) = \frac{1}{2}\left(h(x) + \frac{1}{c}\int_{x_0}^{x} v(\xi)\, d\xi + K\right) \tag{134}$$

K and x_0 are not part of the initial data, so we hope to get rid of them. Fortunately, we are not interested in f and g separately, but only $u(x,t) = f(x-ct) + g(x+ct)$. Indeed, adding f, g, we express the solution of the wave equation entirely in terms of initial height and velocity

$$u(x,t) = \frac{1}{2}\left[h(x-ct) + h(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi\right]. \tag{135}$$

It is instructive to plot a movie of this solution, for instance in the case of zero initial velocity and a simple initial height profile such as $h(x) = e^{-x^2/2}$. One finds two little waves moving away from $x = 0$. The height at x_o at time t_o depends on the initial ($t = 0$) height at points $x_o - ct_o$ and $x_o + ct_o$. So the initial height only at points a distance ct_o from the observation point x_o can affect the height at the point of observation. This indicates that these 'signals' travel at the speed c[6]. The initial velocity $v(x)$ only at points within a distance ct_o from the observation point can affect the height at the observation point.

7 Canonical transformations

7.1 Introduction

• Recall that the space of generalised coordinates and momenta is called phase space. Hamilton's equations $\dot{q}^i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q^i}$ may be easier to solve (or understand qualitatively) in some systems of coordinates and momenta compared to others. For instance, there may be more cyclic coordinates in one system. E.g., for a particle in a central potential $V(r)$ on the plane, the eom are simpler to handle in polar coordinates r, θ than in Cartesian coordinates x, y. From the Lagrangian

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - V\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) = \tilde{L}(r,\theta,\dot{r},\dot{\theta}), \tag{136}$$

θ is a cyclic coordinate and its conjugate momentum $L_z = p_\theta = mr^2\dot{\theta}$ is conserved. On the other hand, neither p_x nor p_y is conserved. We have checked that Hamilton's equations take the same form in cartesian and polar coordinates:

$$\begin{aligned} & \dot{x} = \frac{\partial H}{\partial p_x},\ \dot{y} = \frac{\partial H}{\partial p_y},\ \dot{p}_x = -\frac{\partial H}{\partial x},\ \dot{p}_y = -\frac{\partial H}{\partial y} \quad \text{where} \quad p_x = \frac{\partial L}{\partial \dot{x}} \ \text{ and } \ p_y = \frac{\partial L}{\partial \dot{y}} \\ \Leftrightarrow \quad & \dot{r} = \frac{\partial H}{\partial p_r},\ \dot{\theta} = \frac{\partial H}{\partial p_\theta},\ \dot{p}_r = -\frac{\partial H}{\partial r},\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \quad \text{where} \quad p_r = \frac{\partial L}{\partial \dot{r}} \ \text{ and } \ p_\theta = \frac{\partial L}{\partial \dot{\theta}}. \end{aligned} \tag{137}$$

We say that the transformation from cartesian coordinates and conjugate momenta (x, y, p_x, p_y) to polar coordinates and conjugate momenta $(r, \theta, p_r, p_\theta)$ is a canonical transformation. We also check that the fundamental Poisson brackets among coordinates and momenta are preserved

$$\{x, p_x\} = \{y, p_y\} = 1, \qquad \{x, p_y\} = \{y, p_x\} = \{x, y\} = \{p_x, p_y\} = 0$$

[6]Note that the speed at which these (transverse) signals travel is quite distinct from the instantaneous vertical velocity $\dot{u}$ of a point on the string.

$$\text{and} \qquad \{r, p_r\} = \{\theta, p_\theta\} = 1, \qquad \{r, p_\theta\} = \{\theta, p_r\} = \{r, \theta\} = \{p_r, p_\theta\} = 0. \tag{138}$$

• Suppose we start with a system of coordinates q^i and conjugate momenta p_i, in which Hamilton's equations take the standard form $\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$. A canonical transformation (CT) of coordinates and momenta from old ones (q^i, p_i) to new ones (Q^i, P_i) is one that preserves the form of Hamilton's equations. What use is this concept? At the very least, if we make a change of variables on phase space that is known to be canonical for independent reasons, then we would not need to re-derive the equations of motion, they are guaranteed to take the Hamiltonian form in the new variables. In fact, canonical transformations are a widely useful and deep idea, as we will see.

• But not every choice of coordinates and momenta is canonical. For example, we notice that Hamilton's equations treat coordinates and momenta on a nearly equal footing. So suppose we simply exchange coordinates and momenta by defining $Q = p$ and $P = q$. Then the hamiltonian may be written in terms of the new variables $H(q, p) = H(P, Q) \equiv \tilde{H}(Q, P)$. We find that

$$\dot{Q} = \dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial \tilde{H}}{\partial P} \quad \text{and} \quad \dot{P} = \dot{q} = \frac{\partial H}{\partial p} = \frac{\partial \tilde{H}}{\partial Q}. \tag{139}$$

So the eom in the new variables *do not* have the form of Hamilton's equations, they are off by a sign. So $(q, p) \mapsto (p, q)$ is *not* a canonical transformation. We may also check that the transformation *does not* preserve the fundamental p.b.

$$\{q, p\} = 1 \quad \text{while} \quad \{Q, P\} = \{p, q\} = -1 \tag{140}$$

• Any change of coordinates alone ('point transformation') $q^i \to Q^i$, with the associated 'induced' change in momenta $P_i = \frac{\partial \tilde{L}}{\partial \dot{Q}^i}$ is automatically canonical (provided we have a Lagrangian in mind). An example of such a canonical transformation is the one from cartesian to polar coordinates for a free particle (or one in a potential $V(x, y)$) on a plane. The interesting thing is that there are canonical transformations that are *more general* than those resulting from changes of coordinates (point transformations) on $\mathcal{Q}$. Perhaps the simplest such examples are (1) $Q = p, P = -q$ and (2) $Q = -p, P = q$ which mix coordinates and momenta for one degree of freedom. Check that Hamilton's equations retain their form, as do the fundamental Poisson brackets.

• In the above examples of CTs, along with Hamilton's equations, the fundamental p.b. among coordinates and momenta were also preserved. This is true in general. It is worth noting that a transformation $(q, p) \to (Q, P)$ is canonical irrespective of what the hamiltonian is. The form of Hamilton's equations must be unchanged for *any* smooth $H(q, p)$. Preservation of p.b. allows us to state the condition of canonicity without reference to the hamiltonian.

7.2 Four points of view on canonical transformations

• An invertible and sufficiently differentiable transformation from old canonical variables (q^i, p_j) to a new set of variables $Q^i = Q^i(q^1, \ldots, q^n, p_1, \ldots, p_n), P_j = P_j(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is canonical if any of these conditions is satisfied:

• (1) The Fundamental p.b. between coordinates and momenta is preserved, i.e.,

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0 \tag{141}$$

implies that

$$\{Q^i, P_j\} = \delta^i_j, \quad \{Q^i, Q^j\} = \{P_i, P_j\} = 0. \tag{142}$$

Here all p.b. are evaluated by differentiating with respect to the old variables, in other words, all these p.b are $\{.,.\}_{q,p}$.

• (2) Hamilton's equations take the same form in the new variables

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \Rightarrow \quad \dot{Q}^i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q^i}. \tag{143}$$

where $\tilde{H}(Q,P) = H(q(Q,P), p(Q,P))$ is the hamiltonian re-expressed in the new variables.

• (3) The p.b. of any pair of observables satisfies

$$\{f,g\}_{q,p} = \{f,g\}_{Q,P}. \tag{144}$$

So the formula for p.b. calculated by differentiating with respect to coordinates and momenta is the same in the new variables.

• (4) For one degree of freedom, the signed area element on the phase plane is preserved, i.e., $dqdp = dQdP$. For several degrees of freedom, the area element in every 2-plane in phase space must be preserved (i.e. the symplectic 2-form must be preserved).

• We will not prove the equivalence of these statements here. But let us illustrate them in the case of one degree of freedom.

• (1) $\Leftrightarrow$ (4) For e.g. let us see why preservation of area and fundamental p.b. are the same. The only non-trivial p.b. for 1 dof is $\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$. We notice that this is the expression for the Jacobian determinant for the change of coordinates on phase space

$$\det J = \det \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = 1 \quad \Leftrightarrow \quad \{Q,P\} = 1. \tag{145}$$

Now the 'signed' area element under a change of variables transforms as $dQdP = \det J \; dqdp$[7]. So preservation of the signed area element is the same as preservation of fundamental p.b. By signed area element we mean that the area 'vector' points in the direction of the cross product of infinitesimal vectors along the q and p coordinate directions. A reflection about any axis is orientation reversing and reverses the sign of the area element.

• (3) $\Leftrightarrow$ (1) We try to express the p.b. of two observables $\{f,g\} = \{f,g\}_{qp}$ in terms of $\{f,g\}_{QP}$. A function $f(q,p)$ can be regarded as a function of the new variables by substitution $f(q,p) = f(q(Q,P), p(Q,P))$. We will use subscripts to denote partial derivatives $f_P = \frac{\partial f}{\partial P}$ etc. By the chain rule and rearranging terms,

$$\begin{aligned} \{f,g\} &= f_q g_p - f_p g_q = (f_Q Q_q + f_P P_q)(g_Q Q_p + g_P P_p) - (f_Q Q_p + f_P P_p)(g_Q Q_q + g_P P_q) \\ &= f_Q g_P (Q_q P_p - Q_p P_q) + f_P g_Q (P_q Q_p - P_p Q_q) + f_Q g_Q (Q_q Q_p - Q_p Q_q) + f_P g_P (P_q P_p - P_p P_q) \\ &= (f_Q g_P - f_P g_Q)\{Q,P\} + f_Q g_Q \{Q,Q\} + f_P g_P \{P,P\}. \end{aligned} \tag{146}$$

Of course, the last two terms are identically zero by anti-symmetry of p.b., but we displayed them as they help in writing the corresponding formula for n degrees of freedom:

$$\{f,g\} = \left(f_{Q^i} g_{P_j} - f_{P_j} g_{Q^i}\right)\{Q^i, P_j\} + f_{Q^i} g_{Q^j}\{Q^i, Q^j\} + f_{P_i} g_{P_j}\{P_i, P_j\}. \tag{147}$$

[7] If the Jacobian determinant is new to you, use it to work out the area element in plane polar coordinates starting from Cartesian coordinates $dx\,dy = r\,dr\,d\theta$ and also in spherical polar coordinates $dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Now we see that

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial Q^i}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial Q^i} \right) = \{f,g\}_{Q,P}. \tag{148}$$

iff the new coordinates and momenta satisfy canonical p.b. relations, i.e., if

$$\{Q^i, P_j\} = \delta^i_j, \quad \text{and} \quad \{Q^i, Q^j\} = 0 = \{P_i, P_j\}. \tag{149}$$

Thus a transformation is canonical if the p.b. of any pair of observables is given by the same sort of formula whether computed using the old or new variables:

$$(q,p) \mapsto (Q,P) \quad \text{is a canonical transformation iff} \quad \{f,g\}_{q,p} = \{f,g\}_{Q,P} \quad \forall \ f,g. \tag{150}$$

7.3 Examples of Canonical Transformations

1. The identity $Q = q, P = p$ is a canonical transformation, fundamental p.b. are clearly preserved.

2. Exchange of coordinates and momenta $Q = p, P = q$ is not canonical since $\{Q,P\} = -1$. Such an exchange is orientation reversing, $\det J = -1$

3. Exchange with a sign $Q = -p, P = q$ is a CT, it preserves p.b. The existence of this CT is what one means when one says that hamilton's equations treat coordinates and momenta on a nearly equal footing. There is another exchange $Q = p, P = -q$ which too is a CT.

4. A translation on the phase plane $q \to q + a, p \to p + b$ by the vector (a,b) is a CT. The Jacobian matrix here is the identity and has unit determinant, so areas are preserved.

5. A rotation on the phase plane preserves area elements, it is canonical. The Jacobian here is the rotation matrix given below. It has unit determinant $\cos\theta^2 + \sin\theta^2 = 1$:

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{151}$$

6. Translations and rotations each form a group, two dimensional and one dimensional, parametrized by the vector $(a,b) \in \mathbb{R}^2$ and the angle $\theta \in [0, 2\pi)$. We may compose translations and rotations to form the group of rigid motions of the phase plane (the Euclidean group). All of them are canonical transformations.

7. On the other hand, a reflection such as $Q = -q, P = p$ is not a CT, the sign of the fundamental p.b. is reversed. It also does not preserve the signed area element, it reverses the orientation. We could write symbolically $dQdP = -dqdp$. The determinant of the Jacobian matrix is minus one.

8. A scaling of coordinates and momenta by a real constant $Q = \lambda q, \ P = \lambda p$ is in general not area preserving (except if $\lambda^2 = 1$). On the other hand, the scaling $Q = \lambda q, P = \lambda^{-1} p$ for $\lambda \neq 0$ does preserve areas. It maps squares to rectangles that are thin and tall or short and fat!

9. Time evolution by any hamiltonian gives us important examples of canonical transformations. Recall that the equal time Poisson brackets of coordinates and momenta

$$\{q^i(t), p_j(t)\} = \delta^i_j \quad \text{and} \quad \{q^i(t), q^j(t)\} = \{p_i(t), p_j(t)\} = 0 \tag{152}$$

are valid at all times. So the map from $(q^i(t_1), p_i(t_1))$ to $(q^i(t_2), p_i(t_2))$ which is a map from phase space to itself, is canonical for any times t_1, t_2. So hamiltonian evolution gives us a 1-parameter family of canonical transformations, the parameter is time. Different hamiltonians (say with different potentials $V(q)$, give rise to various 1-parameter families of canonical transformations).

10. Area & orientation preserving maps of the phase plane are all the canonical transformations for one degree of freedom. These include (but are not restricted to) rigid motions like translations and rotations of the *phase plane*. For example, time evolution by a generic hamiltonian is a CT which in general will morph a nice looking disk on the phase plane into a complicated region having the same area. The harmonic oscillator hamiltonian $H = p^2/2m + \frac{1}{2}kq^2$ produces clockwise rotation of the phase plane if $k/2 = 1/2m$. The space of hamiltonians is infinite dimensional, corresponding to various possible choices of potential functions $V(q)$. So we get a large supply of CTs (and area preserving maps of the phase plane) by various choices of hamiltonians.

11. Let us work out one example of a canonical transformation in detail. Consider a free particle on the half line $q > 0$ with equation of motion $m\ddot{q} = 0$ following from the Lagrangian $L(q, \dot{q}) = m\dot{q}^2$ with conjugate momentum $p = m\dot{q}$. Suppose we change coordinates to $Q = q^2$. Then $\dot{Q} = 2q\dot{q}$ and the equation of motion $\ddot{q} = 0$ becomes $\ddot{Q} + \frac{\dot{Q}^2}{2Q} = 0$. This is in fact the EL equation following from the new Lagrangian is $\tilde{L}(Q, \dot{Q}) = \frac{m\dot{Q}^2}{8Q}$. The new momentum is $P = \frac{\partial L}{\partial \dot{Q}} = m\dot{Q}/4Q$. The new variables can be written in terms of the old ones $Q = q^2$ and $P = p/2q$. The p.b. $\{Q, P\} = \{q^2, p/2q\} = 1$ so this transformation is canonical. Changing variables in the old hamiltonian $H = p^2/2m$ gives us the new hamiltonian $\tilde{H}(Q, P) = 2P^2Q/m$. Hamilton's equations that follow, $\dot{Q} = 4QP/m$ and $\dot{P} = -2P^2/m$ imply the same second order equation $\ddot{Q} + \dot{Q}^2/2Q = 0$ as obtained before, showing that the form of hamilton's equations does not change under this transformation. Show that $\tilde{H}$ is the Legendre transform of the new lagrangian $\tilde{L}$. Moreover, the Jacobian matrix for the transformation $Q = q^2, P = p/2q$ is $J = \begin{pmatrix} 2q & 0 \\ -p/2q^2 & 1/2q \end{pmatrix}$ has unit determinant ensuring the preservation of the area element.

7.4 Generating function for infinitesimal canonical transformations

• The condition for a transformation from canonical coordinates and momenta (q_i, p_i) to new ones (Q^i, P_i) to be canonical is that the Poisson brackets must be preserved. It would be nice to find an explicit way of producing canonical transformations. Let us address this question for infinitesimal canonical transformations, those that depart from the identity transformation by a small amount. It turns out that any such canonical transformation can be expressed in terms of a single 'generating' function on phase space. In other words, we consider transformations of the form

$$Q^i = q^i + \delta q^i(q, p) \quad \text{and} \quad P_i = p_i + \delta p_i(q, p) \quad \text{where} \quad \delta q^i, \delta p_i \quad \text{are small.} \tag{153}$$

Note that we do not expand $\delta q, \delta p$ in powers of q and p, we are not assuming that q, p are small.

• We mentioned that time-evolution by any hamiltonian is a CT. Under infinitesimal time evolution $\delta q = \frac{\partial H}{\partial p} = \{q, H\}$ and $\delta p = -\frac{\partial H}{\partial q} = \{p, H\}$. We say that $H(q,p)$ generates infinitesimal time-evolution. Since this is true for any hamilton function H we may say more generally that any observable $f(q,p)$ generates an infinitesimal canonical transformation via the p.b.:

$$\delta q^i = \frac{\partial f}{\partial p_i} = \{q^i, f\} \quad \text{and} \quad \delta p_i = -\frac{\partial f}{\partial q^i} = \{p_i, f\} \tag{154}$$

f is determined up to an additive constant. In the case of $\mathbb{R}^{2n}$ phase space, all infinitesimal CTs may be obtained through appropriate choices of generators $f(q,p)$. It is also possible to build up finite CTs by composing a succession of infinitesimal ones. We will say more about finite CTs later.

• E.g. what infinitesimal CT does the angular momentum component ϵL_z generate? One finds

$$\delta x = -\epsilon y, \delta y = \epsilon x, \delta z = 0 \quad \text{and} \quad \delta p_x = -\epsilon p_y, \delta p_y = \epsilon p_x, \delta p_z = 0. \tag{155}$$

This CT is a counter clockwise rotation in the $x-y$ plane and $p_x - p_y$ plane by the small angle ϵ as we see by writing it as

$$\begin{pmatrix} x+\delta x \\ y+\delta y \\ z+\delta z \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \approx \begin{pmatrix} \cos\epsilon & -\sin\epsilon & 0 \\ \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{for small} \quad \epsilon. \tag{156}$$

A similar matrix representation works for the momenta as well. We say that the components of angular momentum ϵL_i generate counter-clockwise rotations of the position and momentum vectors about the i^{th} axis.

• ϵp_x generates translations in x, $\delta x = \{x, \epsilon p_x\} = \epsilon$ and $\delta y = \delta p_x = 0$ etc.

• It is also interesting to have an expression for the infinitesimal change in a given observable $g(q,p)$ due to a canonical transformation generated by $f(q,p)$:

$$\delta g = \frac{\partial g}{\partial q^i}\delta q^i + \frac{\partial g}{\partial p_i}\delta p_i = \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} = \{g, f\}. \tag{157}$$

So the change in any observable is given by its p.b. with the infinitesimal generator.

7.5 Symmetries & Noether's theorem in the hamiltonian framework

• In the hamiltonian formalism, it is natural to define a symmetry transformation as a canonical transformation $(q^i, p_i) \to (Q^i, P_i)$ that leaves the hamiltonian invariant. The former condition ensures that a symmetry preserves the p.b. This requirement allows us to obtain a conserved quantity from an infinitesimal symmetry. This is expected from Noether's theorem, which we proved in the Lagrangian framework. Symmetries of the hamiltonian that aren't CTs, generally do not lead to conserved quantities.

• E.g., if the hamiltonian is independent of a coordinate q, as for a free particle $H = p^2/2m$, then it is invariant under translations of q, $H(q,p) = H(q+a,p)$. These are implemented by

the p.b. preserving CT $q \to q + a, p \to p$. q is then a cyclic coordinate and the conjugate momentum is conserved $\dot{p} = -\frac{\partial H}{\partial q} = 0$. More generally, a transformation is said to leave the hamiltonian invariant if $H(q,p) = H(Q(q,p), P(q,p))$. The value of energy does not change if we evaluate the hamiltonian at a point on phase space related to the original one by a symmetry transformation. Note that the above translations are a family of symmetries parametrized by a. If a is small, we call it an infinitesimal translation.

• The identity CT $Q^i = q^i, P_i = p_i$ is always a symmetry. Families of symmetries (they form a group in general) may be discrete or continuous. Continuous symmetries are those that may be continuously deformed to the identity. In the case of translations, this is done by taking $a \to 0$. Finite continuous symmetries can be built by composing many infinitesimal ones. We will be concerned with infinitesimal symmetry transformations, which, regarded as CTs, admit an infinitesimal generator $f(q,p)$. E.g. $f(q,p) = \mathbf{p} \cdot \mathbf{a} = p_i a^i$ for a fixed vector $\vec{a}$ generates a translation of coordinates by $\vec{a}$, since

$$\delta q^i = \frac{\partial f}{\partial p^i} = a^i \quad \text{and} \quad \delta p_i = -\frac{\partial f}{\partial q^i} = 0. \tag{158}$$

Now the change in any observable g due to the symmetry transformation generated by f is $\delta g = \{g, f\}$. In particular, since the hamiltonian is invariant under a symmetry, we must have $0 = \delta H = \{H, f\} = 0$. By hamilton's equation this means $\dot{f} = \{f, H\} = 0$. It follows that the generator f of the infinitesimal symmetry, is a constant of motion. Thus we have a Hamiltonian version of Noether's theorem. The symmetry generator is the conserved quantity (Noether's 'charge'). In the above example, it means $p \cdot a$ is a conserved quantity if the hamiltonian is invariant under translations of coordinates by any small vector $\vec{a}$. This means the component of momentum in any direction is conserved for a free particle.

7.6 Brief comparison of classical and quantum mechanical formalisms

• This is a good opportunity to compare certain features of classical and quantum mechanics.

1. In CM, the space of (pure) states is the phase space. Each point in phase space is a possible state. In QM the space of states is the quantum mechanical Hilbert space (vector space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$). Unlike in CM, the space of quantum states satisfies the principle of linear superposition.

2. In CM, observables are smooth real-valued functions on phase space. In QM, observables (A, B etc) are self-adjoint (hermitian) operators on Hilbert space. Self-adjointness is the analogue of reality, both of which ensure that results of measurements are real numbers.

3. The Poisson bracket of observables in CM is replaced by the commutator of operators (upto a factor of $i\hbar$) in QM, e.g. $\{x, p\} = 1 \longrightarrow \frac{1}{i\hbar}[x, p] = 1$. Both operations map a pair of observables to a new observable.

4. Hamilton's equation for the time derivative of an observable f is $\frac{df}{dt} = \{f, H\}$. The quantum mechanical version is the Heisenberg equation of motion $i\hbar \frac{d\hat{f}}{dt} = [\hat{f}, \hat{H}]$.

5. A dynamical variable f that Poisson commutes with the hamiltonian is a classical constant of motion. A quantum mechanical operator $\hat{A}$ that commutes with the hamiltonian $[\hat{A}, \hat{H}] = 0$ is a conserved quantity.

6. In CM, time evolution is a 1-parameter family of canonical transformations. In QM, time evolution is a 1-parameter family of unitary transformations $U(t) = e^{-iHt/\hbar}$.

7. Unitary transformations ($|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$ and $A \to A' = UAU^\dagger$ with $U^\dagger U = I$) are quantum analogs of canonical transformations. Both preserve the structure of the formalism. CTs preserve the fundamental p.b. while unitary transformations preserve the Heisenberg canonical commutation relations, since $[A', B'] = U[A, B]U^\dagger$ and in particular $[q', p'] = U[q, p]U^\dagger = U(i\hbar)U^\dagger = i\hbar$. Unitary transformations also preserve inner products $\langle U\phi|U\psi\rangle = \langle\phi|U^\dagger U\psi\rangle = \langle\phi|\psi\rangle$.

7.7 Liouville's theorem

• We will apply the idea of infinitesimal generator for a CT to establish an interesting theorem of Liouville on the geometric nature of CT. Previously, we saw that for one degree of freedom, CTs preserve areas in phase space. This is a special case of Liouville's theorem. For n degrees of freedom, it says that CTs preserve $2n$-dimensional 'volumes' in phase space. In other words, suppose a $2n$-dimensional region in phase space $D \subset \mathbb{R}^{2n}$ is mapped by a CT to a new region $D' \subset \mathbb{R}^{2n}$. Then $\mathrm{Vol}(D) = \mathrm{Vol}(D')$. Alternatively, it says that the volume element in phase space is invariant under a CT

$$\prod_{i=1}^{n} dQ^i \prod_{j=1}^{n} dP_j \;=\; \prod_{i=1}^{n} dq^i \prod_{j=1}^{n} dp_j. \tag{159}$$

For a general transformation, the determinant of the Jacobian matrix of first partials appears as a pre-factor on the rhs

$$J = \begin{pmatrix} \frac{\partial Q^i}{\partial q^j} & \frac{\partial Q^i}{\partial p_j} \\ \frac{\partial P_i}{\partial q^j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}_{2n\times 2n}, \quad \text{where each sub-matix is an } n\times n \text{ block with } 1 \le i, j \le n. \tag{160}$$

So Liouville's theorem says that $\det J = 1$ for a canonical transformation. Note that unlike for one degree of freedom, for $n > 1$, $\det J = 1$ is *not* a sufficient condition for a transformation to be canonical.

• Let us establish Liouville's theorem for infinitesimal canonical transformations by using our expressions for Q^i, P_j in terms of an infinitesimal generator[8] ϵf

$$Q^i \approx q^i + \epsilon\frac{\partial f}{\partial p_i} \quad \text{and} \quad P_i \approx p_i - \epsilon\frac{\partial f}{\partial q^i} \tag{161}$$

Let us first look at the simple case of $n = 2$ degrees of freedom, where

$$Q^1 \approx q^1 + \epsilon f_{p_1}, \;\; Q^2 \approx q^2 + \epsilon f_{p_2}, \;\; P_1 \approx p_1 - \epsilon f_{q^1} \;\text{ and }\; P_2 \approx p_2 - \epsilon f_{q^2} \tag{162}$$

and sub-scripts denote partial derivatives. In this case the Jacobian matrix

$$J \approx I + \epsilon \begin{pmatrix} f_{q^1p_1} & f_{q^2p_1} & f_{p_1p_1} & f_{p_1p_2} \\ f_{q^1p_2} & f_{q^2p_2} & f_{p_1p_2} & f_{p_2p_2} \\ -f_{q^1q^1} & -f_{q^2q^1} & -f_{p_1q^1} & -f_{p_2q^1} \\ -f_{q^1q^2} & -f_{q^2q^2} & -f_{p_1q^2} & -f_{p_2q^2} \end{pmatrix} = I + \epsilon F \tag{163}$$

[8] ϵ is a small parameter which will help us keep track of infinitesimals, we will ignore quantities of order ϵ^2.

departs from the identity by an infinitesimal matrix of second partials of f.

• Now[9]

$$\det J = \det[I + \epsilon F] = 1 + \epsilon \text{ tr } F + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(\epsilon^2) \tag{164}$$

since F is traceless. So for two degrees of freedom we have shown that an infinitesimal canonical transformation preserves the (4-dimensional) volume element in phase space.

• The case of n-degrees of freedom is analogous. The $2n \times 2n$ Jacobian matrix is made of $n \times n$ blocks

$$J = \begin{pmatrix} \delta_{ij} + \epsilon \frac{\partial^2 f}{\partial p_i \partial q^j} & \epsilon \frac{\partial^2 f}{\partial p_i \partial p_j} \\ -\epsilon \frac{\partial^2 f}{\partial q^i \partial q^j} & \delta_{ij} - \epsilon \frac{\partial^2 f}{\partial q^i \partial p_j} \end{pmatrix} \Rightarrow \det J \approx 1 + \epsilon \sum_{i=1}^{n} \frac{\partial^2 f}{\partial p_i \partial q^i} - \epsilon \sum_{i=1}^{n} \frac{\partial^2 f}{\partial q^i \partial p_i} = 1 \tag{165}$$

Thus, an infinitesimal canonical transformation preserves the volume element in phase space. Synthesizing a finite canonical transformation by composing a succession of N infinitesimal ones and letting $N \to \infty$ and $\epsilon \to 0$, we argue that finite canonical transformations also preserve the phase volume. One needs to show that the terms of order ϵ^2 and higher, will not contribute to the Jacobian of a finite CT.

• In particular, hamiltonian time evolution preserves phase volume. This is true even if the hamiltonian is explicitly time dependent. All we need is for the equations of motion to be expressible in Hamiltonian form $\dot{q}^i = \frac{\partial H}{\partial p_i}, \dot{p}_i = \frac{\partial H}{\partial q^i}$ and this is true even if the Lagrangian depends explicitly on time (see the section on Hamilton's equations). At each instant of time, H generates an infinitesimal CT that preserves the phase volume. Of course, if H is explicitly time-dependent, the CT will change with time, but phase volume will still be preserved. Note that dissipative systems do not admit a standard Lagrangian or hamiltonian description, there is no function $H(q, p, t)$ for which hamilton's equations reproduce the equations of motion. Typically, for dissipative systems, the volume in phase space is a decreasing function of time (e.g. for the damped harmonic oscillator $m\ddot{x} = -kx - \gamma\dot{x}$, irrespective of what initial conditions one considers, the mass comes to rest at the equilibrium point $(x = 0, m\dot{x} = 0)$, so the phase space area shrinks to zero).

• Application to statistical mechanics: Consider the gas molecules in a room, modeled as a system of N classical point particles. The phase space is $6N$ dimensional with coordinates $\vec{q}^1 \cdots \vec{q}^N, \vec{p}_1 \cdots \vec{p}_N$. Now owing to the difficulty of determining the initial values of these variables, we may at best be able to say that the initial conditions lie within a certain region D of phase space, compatible with our rough knowledge of the initial state (say the initial temperature and extent of the room etc.). Each of the phase points in D will evolve in time and trace out a phase trajectory (note that a point in D is not a gas molecule, it is one state of all the gas molecules!). In this manner D itself will evolve to a new region D' which contains the possible phase points at a later time. We are often not interested in locations and momenta of individual gas molecules but average properties of the gas (such as mean pressure or internal energy). These may be obtained by computing an average over the region of phase space D'. Liouvilles's theorem says that this region of phase space evolves in time as an 'incompressible fluid' (retaining its $2n$ dimensional volume). Note that this is true even if the gas itself is compressible! In general, the shape of the region will get distorted with time, while maintaining a constant $6N$-dimensional volume.

[9] Suppose the eigenvalues of $J = I + \epsilon F$ are $\lambda_1, \cdots, \lambda_{2n}$. Then from the characteristic equation $\det(J - \lambda I) = \det(\epsilon F - (\lambda - 1)I) = 0$ we see that the eigenvalues of ϵF are $\lambda_1 - 1, \cdots \lambda_{2n} - 1$. Hence the eigenvalues of F are $f_1 = \frac{\lambda_1 - 1}{\epsilon}, \cdots \frac{\lambda_{2n} - 1}{\epsilon}$. Thus $\det J = \lambda_1 \cdots \lambda_{2n} = (1 + \epsilon f_1) \cdots (1 + \epsilon f_{2n}) = 1 + \epsilon(f_1 + \cdots + f_{2n}) + \mathcal{O}(\epsilon^2) = 1 + \epsilon \text{ tr } F + \mathcal{O}(\epsilon^2)$. Alternatively, assuming the identity $\det J = \exp(\text{ tr } \log(I + \epsilon F))$, one may proceed by expanding in powers of ϵ.

7.8 Variational principles for Hamilton's equations

• We seek an extremum principle for Hamilton's equations, just as we had one for Lagrange's equations: $S[q] = \int L\, dt$ and $\delta S = 0$. Hamilton's variational principle for his equations is given by the functional of a path on *phase space* $(q^i(t), p_j(t))$

$$\mathcal{S}[q,p] = \int_{t_i}^{t_f} \left[p_i \dot{q}^i - H(q,p) \right]\, dt. \tag{166}$$

Recall that $L(q,\dot{q}) = \mathrm{ext}_p(p\dot{q} - H(q,p))$, which motivates the formula for $S[q,p]$. However, here we do not extremize in p. Rather, in the integral, we regard $q(t)$ and $p(t)$ as independent ingredients used to specify the phase path and $\dot{q}$ as obtained by differentiating $q(t)$. Note that $S[q]$ is a functional of a path on configuration space, while $\mathcal{S}[q,p]$ is a functional of a path on phase space. They are not the same, though we call both 'action'. We ask that this functional $\mathcal{S}[q,p]$ be stationary with respect to small variations in the phase path $(q(t), p(t))$ while holding $\delta q(t_i) = 0$ and $\delta q(t_f) = 0$. Note that we do not constrain $\delta p(t_i)$ or $\delta p(t_f)$. That would be an over specification[10]. Now

$$\delta\mathcal{S} = \int_{t_i}^{t_f} \left[\delta p_i\, \dot{q}^i + p_i \delta\dot{q}^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right]\, dt + \ldots \tag{167}$$

We find upon integrating by parts in the second term and using $\delta q(t_{i,f}) = 0$,

$$\mathcal{S}[q+\delta q, p+\delta p] = \mathcal{S}[q,p] + \int_{t_i}^{t_f} \left[\dot{q}^i \delta p_i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right] dt + \ldots. \tag{168}$$

The action must be stationary with respect to arbitrary infinitesimal independent variations δp, δq subject to $\delta q(t_i) = \delta q(t_f) = 0$. So the coefficients of δp and δq must individually vanish. Thus we recover Hamilton's equations at all times $t_i < t < t_f$:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{169}$$

Hamilton's equations treat position and momentum on an equal footing except for a sign. But the above boundary conditions treat them asymmetrically. This is a clue that there is another variational principle for Hamilton's equations. Consider the functional of a path on phase space

$$\tilde{\mathcal{S}}[q,p] = \int_{t_i}^{t_f} \left[-q^j \dot{p}_j - H(q,p) \right] dt \tag{170}$$

which we extremize with respect to small variations $\delta q, \delta p$ while holding $\delta p_j(t_i) = \delta p_j(t_f) = 0$. Then integrating by parts,

$$\begin{aligned} \delta\tilde{\mathcal{S}} &= \int_{t_i}^{t_f} \left[-\dot{p}_j \delta q^j - q^j \delta\dot{p}_j - \frac{\partial H}{\partial q^j}\delta q^j - \frac{\partial H}{\partial p_j}\delta p_j \right] dt \\ &= \left[\dot{q}^j \delta p_j \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} \left[\left(\dot{p}_j + \frac{\partial H}{\partial q^j} \right) \delta q^j + \left(\dot{q}^j - \frac{\partial H}{\partial p_j} \right) \delta p_j \right] dt. \end{aligned} \tag{171}$$

So $\delta\tilde{\mathcal{S}} = 0$ also implies Hamilton's equations. We will exploit both these variational principles to find generating functions for finite canonical transformations. The utility of the second variational principle is mostly conceptual. In practice, we are rarely interested in finding trajectories connecting specified initial and final momenta.

[10] There would in general not be *any* trajectory joining specified values of q *and* p at *both* t_i and t_f. Demonstrate this in the case of a free particle.

7.9 Generating functions for finite canonical transformations from variational principles

• Transformations between different sets of canonical coordinates and momenta are called canonical transformations. Here we seek to express finite canonical transformations in terms of generating functions. We have already done this for infinitesimal canonical transformations. To do so, we will use Hamilton's variational principle for his equations. Consider the (possibly explicitly time-dependent) map from $(q^i, p_j) \mapsto (Q_i, P_j)$ with the equations of transformation given by the functions

$$Q^i = Q^i(q, p, t) \quad \text{and} \quad P_i = P_i(q, p, t) \tag{172}$$

Such a change is canonical provided there is a new Hamiltonian $K(Q, P, t)$ (previously called $\tilde{H}$) such that the eom in the new variables take the same form as those in the old variables, i.e.,

$$\dot{Q}^i = \frac{\partial K}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial K}{\partial Q^i} \quad \text{while} \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{173}$$

When the transformation is not explicitly dependent on time, $K(Q, P)$ is got by expressing q, p in terms of Q, P in the old Hamiltonian $H(q, p)$. We will see that essentially the same thing continues to be true, but with a slight modification. Now both these sets of Hamilton equations should be equivalent in the sense that if we express Q and P in terms of q and p in the second set, they should reduce to the old Hamilton equations.

• Each set of Hamilton's equations follows from a variational principle:

$$\delta \int_{t_i}^{t_f} \left[p_i \dot{q}^i - H(q, p) \right] dt = 0 \quad \text{and} \quad \delta \int_{t_i}^{t_f} \left[P_i \dot{Q}^i - K(Q, P) \right] dt = 0. \tag{174}$$

The extrema of these two functionals are the same equations (just in different coordinates). One way for this to happen is for the integrands to be the same. But there is also a more general way for this to happen, the integrands could differ by the total time derivative of a function $F_1(q, Q, t)$. Let us see why. Subtracting, we find that the condition for the functional

$$I[q, p, Q, P] = \int_{t_i}^{t_f} \left(p\dot{q} - H - P\dot{Q} + K \right) dt \tag{175}$$

to be extremal is identically satisfied, since it is the difference between two equivalent sets of equations. So this integral must be a constant functional with respect to variations of q, p, Q, P subject to the boundary conditions $\delta q(t_i) = \delta q(t_f) = \delta Q(t_i) = \delta Q(t_f) = 0$. A way for this to happen is for the integrand to be a total time derivative of a function $\dot{F}_1(q, Q, t)$. For, then

$$I = \int_{t_i}^{t_f} \dot{F}_1 \, dt = F_1(q(t_f), Q(t_f), t_f) - F_1(q(t_i), Q(t_i), t_i). \tag{176}$$

And I is then a constant since q and Q are held fixed at the fixed times t_i and t_f. Note that F_1 cannot be taken as a function of p or P since $\delta p(t_i), \delta p(t_f), \delta P(t_i), \delta P(t_f)$ are unconstrained in Hamilton's variational principle and the total derivative of such a term would violate the constancy of I. In other words, a way by which the equations in both old and new variables take the hamiltonian form is for the relation

$$p_i \dot{q}^i - H = P_i \dot{Q}^i - K + \frac{dF_1}{dt}, \tag{177}$$

to hold for some function $F_1(q,Q,t)$. Multiplying through by dt we get

$$pdq - Hdt = PdQ - Kdt + \frac{dF_1}{dt}dt. \tag{178}$$

That the independent variables in F_1 are q, Q, t is also consistent with the fact that the independent differentials appearing in the rest of the terms above are dt, dq, dQ. So as an equation among the independent differentials dq, dQ, dt we have

$$pdq - Hdt = PdQ - Kdt + \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial Q}dQ + \frac{\partial F_1}{\partial t}dt. \tag{179}$$

Comparing coefficients, we read off the relations

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q} \quad \text{and} \quad K(Q,P,t) = H(q,p) + \frac{\partial F_1(q,Q,t)}{\partial t}. \tag{180}$$

$F_1(q,Q)$ is called the generator of the CT. The first two equations determine the equations of transformation. The first may be solved to find $Q = Q(q,p,t)$ and using it, the second expresses $P = P(q,p,t)$. The last relation fixes the new hamiltonian in terms of the old one and the generator. If F_1 does not depend explicitly on time, then it just says that $K(Q,P) = H(q(Q,P), p(Q,P)) = \tilde{H}(Q,P)$ as before. But in general, the new and old hamiltonians differ by the partial time derivative of the generator.

• Not every function $F_1(q,Q,t)$ is a legitimate generator. E.g., $F_1(q,Q) = q + Q$ would imply $p = 1$ and $P = -1$ which in general cannot be solved to express Q, P in terms of q, p. Similarly, $F_1 = q^2 + Q^2$ also does not generate a CT since it implies $p = 2q, P = -2Q$ which cannot be solved to express Q, P as functions of q, p. On the other hand, a choice that does generate a CT is $F_1(q,Q) = qQ$, in which case, $Q = p$ and $P = -q$ exchanges coordinates and momenta up to a sign. What CT does $F_1 = -qQ$ generate?

• In general, for $F_1(q,Q)$ to generate a CT, we need the 'hessian' of unlike second partials $\frac{\partial^2 F_1}{\partial q \partial Q}$ to be non-vanishing This will allow us to use $p = \frac{\partial F_1(q,Q)}{\partial q}$ to solve for Q in terms of q, p, at least locally. When the second partial is non-vanishing $\frac{\partial F_1(q,Q)}{\partial q}$ depends non-trivially on Q which can then be solved for and then inserted in $P = -\frac{\partial F_1(q,Q)}{\partial Q}$ to express $P = P(q,p)$.

• The generator of a finite CT $F_1(q,Q,t)$ is distinct from the infinitesimal generator $f(q,p)$ encountered before. Unlike $f(q,p)$, which generates all infinitesimal CTs, $F_1(q,Q,t)$ does not generate all finite CTs. In particular, the identity transformation $Q = q, P = p$ is not expressible via a generating function $F_1(q,Q,t)$. The latter expresses $p = \frac{\partial F_1(q,Q)}{\partial q} = p(q,Q)$ and $P = -\frac{\partial F_1(q,Q)}{\partial Q} = P(q,Q)$. But for the identity transformation, it is not possible to express P as a function of Q and q. Roughly, $F_1 = qQ + \epsilon g(q,Q)$ is good at generating CTs that are in the vicinity of the one that exchanges coordinates and momenta upto a sign $Q = p,\ P = -q$. It is not a good way of generating CTs in the vicinity of the identity transformation. [Nevertheless, it is possible to get arbitrarily close to the identity CT using a generator of type I, see problems.]

• To find a generator for other canonical transformations, we make use of the second variational principle $\tilde{\mathcal{S}}[Q,P]$ for Hamilton's equations. Here the momenta are held fixed at the end points $\delta P(t_i) = \delta P(t_f) = 0$. For the old hamilton equations, we use the first variational principle $\mathcal{S}[q,p]$ where $\delta q(t_i) = \delta q(t_f) = 0$:

$$\delta \int_{t_i}^{t_f} [p\dot{q} - H(q,p)]\, dt = 0 \quad \text{and} \quad \delta \int_{t_i}^{t_f} [-Q\dot{P} - K(Q,P)]\, dt = 0. \tag{181}$$

These two variational principles give the same equations even if the integrands differ by the total time derivative of a function $F_2(q,P,t)$ since $\delta q, \delta P$ are held fixed at the end points. So we must have

$$pdq - Hdt = -QdP - Kdt + \frac{\partial F_2}{\partial q}dq + \frac{\partial F_2}{\partial P}dP + \frac{\partial F_2}{\partial t}\,dt \tag{182}$$

Thus $F_2(q,P)$ generates a CT, with the equations of transformation given by

$$p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} \quad \text{and} \quad K = H + \frac{\partial F_2}{\partial t}. \tag{183}$$

• It is easily seen that if $F_2(q,P) = qP$, then the resulting transformation is the identity $Q = q, p = P$. In the absence of explicit time dependence, $F_2(q,P)$ is sometimes denoted $W(q,P)$. The above arguments show that F_2 generates a CT and must therefore preserve Poisson brackets.

• The difference between the generating functions $F_1(q,Q)$ and $F_2(q,P)$ lies in the independent variables they depend on. As we have seen, $F_1(q,Q)$ cannot be used to get the identity transformation and one checks that $F_2(q,P)$ cannot be used to get the exchange transformation $Q = p, P = -q$. But there are many CTs that may be generated by both a generating function $F_1(q,Q)$ and one of type $F_2(q,P)$ (we will give non-trivial examples in the context of the harmonic oscillator). In these cases, one wonders whether F_1 and F_2 are related by a Legendre transform, as they produce the same CT. From the difference of the above two relations among differentials,

$$pdq - Hdt = PdQ - Kdt + dF_1 \quad \text{and} \quad pdq - Hdt = -QdP - Kdt + dF_2 \tag{184}$$

we get

$$-QdP + dF_2 = PdQ + dF_1 \;\; \Rightarrow \;\; dF_2(q,P) = d\left[F_1(q,Q) + QP\right] \quad \text{where} \quad P = -\frac{\partial F_1}{\partial Q}. \tag{185}$$

In other words, up to an additive constant, $F_2 = QP + F_1$ with P given as above, or in short,

$$F_2(q,P,t) = \mathrm{ext}_Q\left[QP + F_1(q,Q,t)\right]. \tag{186}$$

• We may obtain two more types of generators $F_3(p,Q,t)$ and $F_4(p,P,t)$ for finite canonical transformations by suitable choices of variational principles for the old and new Hamilton equations.

$$\tilde{\mathcal{S}}[q,p] \;\; \& \;\; \mathcal{S}[Q,P] \;\; \Longrightarrow \;\; F_3(p,Q) \quad \text{while} \quad \tilde{\mathcal{S}}[q,p] \;\; \& \;\; \tilde{\mathcal{S}}[Q,P] \;\; \Longrightarrow \;\; F_4(p,P) \tag{187}$$

• **One wonders if there are generating functions $F_5(q,p)$ and $F_6(Q,P)$ for finite CTs.** The above variational approach doesn't lead to such generators. In Hamilton's action principle, both q and p cannot be held fixed at the end points, so the total time derivative of F_5 would non-trivially modify the form of hamilton's equations and not lead to a CT. Similarly, a generator $F_6(Q,P)$ is also disallowed.

• **Example:** We began our study of canonical transformations with coordinate changes $Q^i(q)$ on configuration space ('point' transformations). The identity is included among such transformations. So let us look for a generator of type $W(q,P)$ that effects a change of coordinates on

$\mathcal{Q}$, for simplicity when $n = 1$. We must have $Q = \frac{\partial W}{\partial P}$ and $p = \frac{\partial W}{\partial q}$. The first equation then implies

$$W(q, P) = PQ(q) + g(q) \tag{188}$$

for some function $g(q)$ of the old coordinates alone. Then $p = PQ'(q) + g'(q)$ or $P = (p - g'(q))/Q'(q)$. This determines the new momentum. A CT that effects a change of coordinates on $\mathcal{Q}$ is clearly not unique, $g(q)$ being an arbitrary function. Different functions $g(q)$ produce different possible new momenta. In our earlier discussion, the new momenta were determined using a Lagrangian. Specification of a Lagrangian ($L(q, \dot{q})$ with a particular dependence on velocities), which 'induces' a change in momenta $P = \frac{\partial \tilde{L}}{\partial \dot{Q}}$ where $\tilde{L}(Q, \dot{Q}) = L(q(Q), \dot{q}(Q, \dot{Q}))$, is like selecting a specific function g. Of course, the simplest possibility is to take $g = 0$, which we will see below corresponds to a Lagrangian with the standard kinetic terms.

• Let us illustrate with the example of the 'point' transformation from cartesian to plane polar coordinates on configuration space. The old coordinates and momenta are x, y, p_x, p_y and the new coordinates and momenta are $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x), p_r, p_\theta$ with p_r, p_θ yet to be determined. By the above arguments, the simplest generating function of the second type, that should take cartesian to plane polar coordinates is one with $g = 0$:

$$W(x, y, p_r, p_\theta) = Q^i(q)P_i = r(x, y)p_r + \theta(x, y)p_\theta = \sqrt{x^2 + y^2}\, p_r + \arctan\left(\frac{y}{x}\right) p_\theta \tag{189}$$

The new coordinates are given by partial derivatives of W and satisfy the defining relations as expected:

$$r = \frac{\partial W}{\partial p_r} = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \frac{\partial W}{\partial p_\theta} = \arctan\left(\frac{y}{x}\right). \tag{190}$$

The old momenta are given by the following partial derivatives of W

$$p_x = \frac{\partial W}{\partial x} = \frac{x}{r}p_r - \frac{y}{r^2}p_\theta \quad \text{and} \quad p_y = \frac{\partial W}{\partial y} = \frac{y}{r}p_r + \frac{x}{r^2}p_\theta. \tag{191}$$

We may invert these relations and express p_r and p_θ in terms of the old coordinates and momenta

$$p_r = \frac{x}{r}p_x + \frac{y}{r}p_y = p_x \cos\theta + p_y \sin\theta = \mathbf{p} \cdot \hat{r} \quad \text{and} \quad p_\theta = xp_y - yp_x = L_z. \tag{192}$$

We see that p_r and p_θ are the familiar radial and angular momenta. So our generating function reproduces the usual conjugate momenta that we derived using the standard Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$. On the other hand, if we took

$$W(x, y, p_r, p_\theta) = rp_r + \theta p_\theta + g(x, y) \quad \text{for} \quad g \neq 0, \tag{193}$$

then the resulting transformation would still be a CT, but the momenta $p_x = \frac{x}{r}p_r - \frac{y}{r^2}p_\theta + \frac{\partial g}{\partial x}$ and $p_y = \frac{\partial W}{\partial y} = \frac{y}{r}p_r + \frac{x}{r^2}p_\theta + \frac{\partial g}{\partial y}$, would not be the usual ones (arising from the above Lagrangian).

7.10 Hamilton Jacobi equation

• Canonical transformations from one canonical set of coordinates and momenta to a more convenient set allow us to better understand the dynamics of a system. The more cyclic coordinates there are, the more the number of conserved conjugate momenta. An extreme set

of canonical variables are those in which the hamiltonian is independent of all coordinates as well as momenta, i.e., if the hamiltonian is a constant. By a choice of zero of energy, this constant may be taken as zero. Now if the hamiltonian in the new variables $K = 0$, then the new coordinates and momenta are both constant in time, and are therefore determined by their initial values: $Q_i(t) = Q_i(0) = \beta_i$ and $P_i(t) = P_i(0) = \alpha_i$. Time evolution is very simple in such variables! All the complications are put in the canonical transformation from the original variables to the new ones! However, it is not always possible to find canonical variables in which $K = 0$. But if it is possible, then the generator of the canonical transformation to such variables must satisfy an interesting first order PDE called the (time-dependent) Hamilton-Jacobi equation. Let us look for a generating function of the second type $F_2(q, P, t)$ for the transformation from $(q, p, H) \to (Q, P, K)$. For reasons to be clarified below, it is conventional in this context to denote F_2 by $S(q, P, t)$ and call it Hamilton's principal function. P_i are the new *constant* momenta and $p = \frac{\partial S}{\partial q}$. Then S must satisfy

$$K = H(q, p, t) + \frac{\partial S(q, P, t)}{\partial t} = 0 \qquad \text{or} \qquad H\left(q_i, \frac{\partial S}{\partial q_j}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{194}$$

This is the Hamilton-Jacobi equation (HJ), a first order (generally non-linear) PDE for the unknown generating function S in $n+1$ variables $q^1, \cdots, q^n, t$. For a particle in a 1D potential $V(q)$, it is a PDE for one unknown function S of two independent variables q, t:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = 0. \tag{195}$$

One is interested in the so-called 'complete integrals/solutions' of HJ, which depend on $n+1$ constants of integration. These solutions are of the form

$$S = S(q^1, \cdots q^n, \alpha_1 \cdots \alpha_n, t). \tag{196}$$

We haven't indicated the dependence on the $(n+1)^{\text{th}}$ constant of integration α_{n+1}. α_{n+1} may be taken as an additive constant in S, since only derivatives of S appear in the HJ eqn. We will choose $\alpha_{n+1} = 0$ since it will be seen not to enter the equations of transformation $p = \frac{\partial S}{\partial q}$, $Q = \frac{\partial S}{\partial P}$ and $K = H + \frac{\partial S}{\partial t}$. The origin of these constants of integration will be clarified when we discuss the method of separation of variables to solve the HJ equation. In favorable cases (such as the free particle), the HJ PDE can be reduced to a set of n *decoupled first order* ODEs, whose solution introduces the required constants of integration.

• The virtue of a 'complete' solution of the HJ equation is that it provides a way of solving for the time evolution of the original mechanical system, i.e., of expressing $q^i(t)$ and $p_j(t)$ in terms of their initial values. First, we are free to take (i.e., *define*) the new constant momenta to equal the constants of integration, i.e., $P_j = \alpha_j$. (We could also take the P_j to be some n independent functions of the α_j.)

• The equations of transformation in terms of the generator S read

$$p_j = \frac{\partial S(q, \alpha, t)}{\partial q^j} \quad \text{and} \quad \beta^i = Q^i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}. \tag{197}$$

The second equation may be used to solve for $q^i = q^i(\alpha_j, \beta^k, t)$. This may be put in the first equation to find $p_i = p_i(\alpha_j, \beta^k, t)$. Now the solution of the mechanical problem in the sense

mentioned above would be obtained if we express α_j, β^k in terms of the initial values of the old variables $q^i(0)$ and $p_i(0)$. To do this, let us consider these equations at $t = 0$. We get

$$p_j(0) = \frac{\partial S(q, \alpha, t)}{\partial q^j} \quad \text{and} \quad \beta^i = \frac{\partial S(q(t), \alpha, t)}{\partial \alpha_i} \quad \text{evaluated at } t = 0. \tag{198}$$

We may use the first equation to express α_i in terms of $q^i(0)$ and $p_i(0)$. Then the second equation gives us β^i in terms of $q^j(0)$ and $p_k(0)$.

• Using the above results, we get

$$q^i(t) = q^i(q^j(0), p_k(0), t) = q^i(\alpha, \beta, t) \quad \text{and} \quad p_i(t) = p_i(q^j(0), p_j(0), t) = p_i(\alpha, \beta, t) \tag{199}$$

These give the solution to the mechanical problem since they express the old coordinates and momenta in terms of their initial values.

• It is not always possible to find a complete solution of the HJ equation. Sometimes, one may find a solution S depending on less than $n + 1$ constants of integration. Even this can be used to provide a partial understanding of the original mechanical problem.

• **Time-independent HJ equation by separation of the time variable:** As we will show shortly, the HJ equation is the classical limit of the time-dependent Schrödinger equation (SE) for the quantum mechanical wave function $\Psi(q, t) = e^{iS(q,t)/\hbar}$ in the $\hbar \to 0$ limit. Recall that for a time-independent hamiltonian, we solve the SE by separating the time dependence from spatial dependence via the product ansatz $\Psi(q, t) = \psi(q)T(t)$. One finds $T(t) = e^{-iEt/\hbar}$ where the separation constant E is the energy eigenvalue in the time-independent Schrodinger equation $H\psi = E\psi$. This suggests that we may separate variables in the HJ equation as well. Since the wave function is the exponential of S, the classical analogue of the product ansatz is an additive ansatz $S(q, t) = W(q) + \Theta(t)$. $\dot{\Theta}(t) = -E$ must be must be equal to a separation constant E with dimensions of energy for this to satisfy the HJ equation. So if $H = H(q, p)$ is not explicitly dependent on time, so that the hamiltonian is a constant of motion, we may separate the coordinate dependence of S from its time-dependence and seek a solution of the HJ equation $H(q, \frac{\partial S}{\partial q}) + \frac{\partial S}{\partial t} = 0$ in the form

$$S(q, \alpha, t) = W(q, \alpha) - Et \tag{200}$$

Inserting this in HJ, we find that it is satisfied provided W solves the time-independent HJ equation for Hamilton's characteristic function

$$E = H\left(q, \frac{\partial W}{\partial q}\right). \tag{201}$$

For a particle in a 1d potential, this is the non-linear ODE $\frac{1}{2m}W'(q)^2 + V(q) = E$.

• **Why do we use the letter S to denote the generating function F_2 in the above discussion?** An interpretation of S is got by computing its time derivative.

$$\dot{S} = \frac{\partial S}{\partial q}\dot{q} + \frac{\partial S}{\partial t} = p_i\dot{q}_i + \frac{\partial S}{\partial t} = p\dot{q} - H \quad \Rightarrow \quad S(t) - S(t_0) = \int_{t_0}^{t} [p\dot{q} - H]dt' \tag{202}$$

So S is the action that appears in Hamilton's variational principle, now regarded as a function of the final time.

7.11 Hamilton-Jacobi equation as semi-classical limit of Schrodinger equation

• We began with Newton's formulation of a mechanical system in terms of a system of non-linear ODEs for cartesian coordinates. We progressed to Lagrange's equations which are still ODEs, but whose form is invariant under changes of coordinates on configuration space. Then came hamilton's ODEs which are form-invariant under canonical transformations on phase space. The Poisson bracket formulation of hamilton's equations $\dot{f} = \{f, H\}$ take the same form for any observable and any system of coordinates on phase space (canonical or not). Now we have reformulated time-evolution of a hamiltonian system in terms of a single non-linear first order PDE for a generating function $S(q, P, t)$. This brings the equations of particle mechanics closer in spirit to the PDEs for waves: classical EM waves in the short wavelength Eikonal approximation and quantum matter waves in the semi-classical approximation. Recall the Schrodinger equation for time evolution of the wave function of a particle in a potential V:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi \tag{203}$$

As we know from the free particle stationary state wave function $\Psi(x,t) = e^{i(px-Et)/\hbar}$, which has an essential singularity at $\hbar = 0$, the wave function itself does not have a good classical limit. But the quantity S defined by $\Psi = e^{iS/\hbar}$ is better placed to have a finite $\hbar \to 0$ limit. We have

$$\nabla\Psi = \frac{i}{\hbar}\Psi\nabla S, \quad \nabla^2\Psi = -\frac{1}{\hbar^2}\Psi\nabla S\cdot\nabla S + \frac{i}{\hbar}\Psi\nabla^2 S, \tag{204}$$

so the Schrodinger equation becomes, upon cancelling $e^{iS/\hbar} \neq 0$,

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}|\nabla S|^2 + V - \frac{i\hbar}{2m}\nabla^2 S \tag{205}$$

No approximation has been made, though we assume that ψ is expressible as $e^{iS/\hbar}$ for some S[11]. In the limit $\hbar \to 0$ we ignore the last term and get the Hamilton-Jacobi evolution equation:

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V = 0 \quad \text{or} \quad \frac{\partial S}{\partial t} + H(q, \nabla S) = 0. \tag{206}$$

• Similarly, show that the time-independent HJ equation is the $\hbar \to 0$ limit of the time-independent Schrödinger equation.

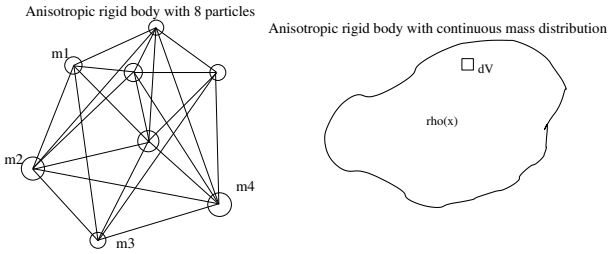
8 Rigid body mechanics

8.1 Lab and co-rotating frames

• A rigid body in mechanics is a system of particles such that the distances between the particles is fixed. E.g. four mass points $m_a, a = 1, 2, 3, 4$ arranged at the vertices of a regular tetrahedron and pairwise connected by light rigid rods. A continuous distribution of mass is also possible, as in a stone. In this case, the discrete index a is replaced with the continuous location index x and the masses of individual particles is replaced by the mass $dm(x) = \rho(x)dV$ in any elemental volume dV around x where $\rho(x)$ is the local mass density. Examples of rigid bodies include a

[11] S would have to diverge at points where Ψ vanishes.

point mass, a pair of point masses connected by a massless rod, a spherical shell, a boomerang, a top, a spaceship, a gyroscope, a tennis ball, a plate, rotational dynamics (ignoring vibrational modes) of molecules in quantum mechanics, etc. Our approach is based on the treatment of Landau and Lifshitz.



• For example, a top fixed at a point on the ground in the uniform gravitational field of the earth executes a motion which involves rotation about its own axis, precession about a vertical axis parallel to the gravitational field and a wobble of the axis of rotation, called nutation. The three periodic motions have their own periods, which if incommensurate prevent the top from returning to its position under time evolution.

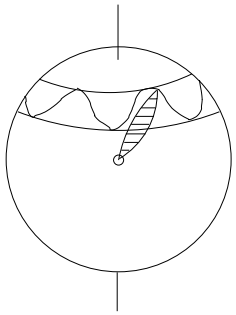


Figure 3: Three periodic motions of a top.

• The motion of points in the rigid body may be described using the lab ('inertial' or 'space' or 'fixed') frame. We will use a system of cartesian co-ordinates X, Y, Z for the lab frame. The radius vector of the point in the body labelled a, relative to the origin of the lab frame, is called $\boldsymbol{r}_a$. The components of $\boldsymbol{r}_a$ relative to the lab axes are $\boldsymbol{r}_a = (X_a, Y_a, Z_a)$. The lab frame coordinates $X_a(t), Y_a(t), Z_a(t)$ (or coordinates in 'space') of a point in the body, in general, change as the body moves. The centre of mass of the body is the point $\mathbf{R} = (\bar{X}, \bar{Y}, \bar{Z})$ whose coordinates are

$$\bar{X} = \frac{1}{M}\sum m_a X_a, \;\; \bar{Y} = \frac{1}{M}\sum m_a Y_a, \;\; \bar{Z} = \frac{1}{M}\sum m_a Z_a, \quad \text{where} \quad M = \sum_a m_a. \tag{207}$$

The sum on a is over all the mass points that make up the body whose total mass is M. The center of mass (CM) is a distinguished point associated to a rigid body (however, it need not lie inside the body!). So it is convenient to consider the location of a mass point labelled a, relative to the center of mass. The corresponding radius vector relative to the CM is called $\mathbf{r}_a$. The components of $\mathbf{r}_a$ relative to the lab axes, in general change with time. The radius vector of the center of mass, relative to the center of mass is obviously the zero vector: $\sum_a m_a \mathbf{r}_a = 0$.

• We may also use a moving system of cartesian coordinates $r_1 = x, r_2 = y, r_3 = z$ referred to a so-called co-rotating frame whose axes are rigidly fixed in the body and participate in the motion. The origin O, of the co-rotating frame is most conveniently chosen to lie at the centre of mass of the body, so $\sum_a m_a x_a = \sum_a m_a y_a = \sum_a m_a z_a = 0$. The co-moving frame components (x_a, y_a, z_a) of the radius vector $\mathbf{r}_a$ of a point in the body are independent of time.

• A point mass moving in 3d space has three degrees of freedom and configuration space $\mathbb{R}^3$.

• If a rigid body is concentrated along a line (e.g. a pair of mass points or a thin wire/pen), the number of degrees of freedom is five. Such a rigid body is called a rigid rotator. A rigid rotator has three translational degrees of freedom, which may be regarded as specifying the location of the center of mass. Once the location of the CM has been fixed, the orientation of the pen is determined by the point on a unit sphere (centered at the CM), at which the line from the CM to the nib intersects it. Thus, the configuration space of a rigid rotator is $\mathbb{R}^3 \times S^2$. The unit 2-sphere is defined as $S^2 = \{\vec{x} \in \mathbb{R}^3 \text{ such that } ||\vec{x}|| = 1\}$.

• A general (i.e. non-collinear) rigid body has six degrees of freedom. We need three coordinates to locate the position of the centre of mass, which we denote $\mathbf{R} = (\bar{X}, \bar{Y}, \bar{Z})$. We are then free to rotate the rigid body about its center of mass in any manner. The orientation of the moving frame relative to the lab frame is determined by such a rotation of 3D space. A rotation is determined by an axis and an angle of rotation. We need two angles to define an axis and one angle to specify the amount of rotation. So rotations are a three parameter family. Thus, a general rigid body has three translational and three rotational degrees of freedom. Moreover, rotations may be composed and inverted; they form the group of special orthogonal matrices. Thus, the configuration space of the rigid body is $\mathbb{R}^3 \times \mathrm{SO}(3)$.

8.2 Infinitesimal displacement and angular velocity of rigid body

• We begin with some kinematical aspects and in particular, the concept of angular velocity. An infinitesimal displacement of a rigid body in a time dt may be expressed as a sum of an infinitesimal translation of the center of mass to its final location (keeping the orientation of the body fixed) and an infinitesimal rotation about the center of mass O that orients the moving frame appropriately. Moreover, any infinitesimal rotation about O is a rotation about some axis passing through O.

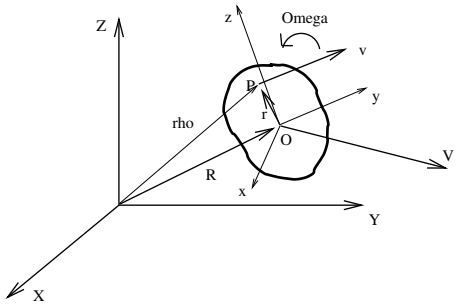


Figure 4: Lab and co-rotating frames. 'rho' corresponds to $\boldsymbol{\rho}$ in the text.

• Let us denote by $\boldsymbol{\rho}$ the radius vector of a point P in the rigid body, relative to the origin of the lab frame. Suppose the same point has the (fixed) radius vector $\mathbf{r}$ relative to the CM. Then

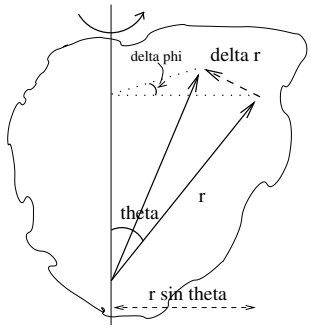


Figure 5: Infinitesimal displacement due to a rotation

$\boldsymbol{\varrho} = \mathbf{R} + \mathbf{r}$ where $\mathbf{R}$ is the radius vector of the centre of mass relative to the origin of the lab frame. We will regard each of these three vectors as given by their components with respect to the lab axes. Now, a small displacement $\delta\boldsymbol{\varrho}$ of P may be written as

$$\delta\boldsymbol{\varrho} = \delta\mathbf{R} + \vec{\delta}\phi \times \mathbf{r} \tag{208}$$

Here $\delta\mathbf{R}$ is the displacement of the center of mass. $\vec{\delta}\phi \times \mathbf{r}$ is the infinitesimal change in the radius vector $\mathbf{r}$ (relative to the lab frame) due to a counter-clockwise rotation about the axis $\vec{\delta}\phi$ by an angle $\delta\phi = |\delta\phi|$. To see why, consult fig. 5. The change in $\mathbf{r}$ under an infinitesimal rotation by angle $\delta\phi$ about the axis $\vec{\delta}\phi$ is in magnitude

$$|\delta\mathbf{r}| = |\mathbf{r}| \ \ \sin\theta \ \ |\delta\phi| \tag{209}$$

$\delta\mathbf{r}$ is perpendicular to both $\mathbf{r}$ and $\vec{\delta}\phi$. So $\delta\mathbf{r} = \vec{\delta}\phi \times \mathbf{r}$.

• Thus dividing by the time δt in which the infinitesimal motion took place,

$$\frac{\delta\boldsymbol{\varrho}}{\delta t} = \frac{\delta\mathbf{R}}{\delta t} + \frac{\vec{\delta}\phi}{\delta t} \times \mathbf{r} \tag{210}$$

Now we let $\delta t \to 0$. If we denote the velocity vector of P relative to the origin of the lab frame by $\boldsymbol{\upsilon} = \frac{d\boldsymbol{\varrho}}{dt}$, the translational velocity of the center of mass by $\mathbf{V} = \frac{d\mathbf{R}}{dt}$ and the angular velocity by $\boldsymbol{\Omega} = \frac{d\vec{\phi}}{dt}$ then we have

$$\boldsymbol{\upsilon} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r} \quad \text{or} \quad \boldsymbol{\upsilon} = \mathbf{V} + \mathbf{v} \quad \text{where} \quad \mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}. \tag{211}$$

$\boldsymbol{\Omega}$ points along the axis of rotation passing through the center of mass (direction of $\boldsymbol{\Omega}$ is along axis $\vec{\delta}\phi$ of right-handed rotation). $\boldsymbol{\Omega}$ may of course change with time both in direction and magnitude.

• To summarize, for each mass point m_a in the rigid body we use the following notation (not quite that of L & L). $\mathbf{R}$ is the radius vector of the center of mass relative to origin of lab frame, and $\mathbf{r}_a$ the radius vector of the point a relative to the center of mass,

$$\boldsymbol{\varrho}_a = \mathbf{R} + \mathbf{r}_a, \quad \text{and differentiating} \quad \boldsymbol{\upsilon}_a = \mathbf{V} + \mathbf{v}_a \quad \text{where} \quad \mathbf{V} = \dot{\mathbf{R}} \quad \text{and} \quad \mathbf{v}_a = \dot{\mathbf{r}}_a = \boldsymbol{\Omega} \times \mathbf{r}_a. \tag{212}$$

$\mathbf{v}_a = \dot{\mathbf{r}}_a$ is the velocity relative to the center of mass. Multiplying by the mass m_a we get

$$\boldsymbol{\pi}_a = m_a\mathbf{V} + \mathbf{p}_a \quad \text{where} \quad \mathbf{p}_a = m_a\mathbf{v}_a = m_a\boldsymbol{\Omega} \times \mathbf{r}_a. \tag{213}$$

The vector sum of all the momenta in the lab frame coincides with the center of mass momentum

$$\mathbf{P} = \sum_a \not{p}_a = M\mathbf{V} + \mathbf{\Omega} \times \sum_a m_a \mathbf{r}_a = M\mathbf{V}. \tag{214}$$

The second sum vanishes as the center of mass lies at the origin of the co-moving frame.

• The coordinates and momenta $(\not{r}_a, \not{p}_a)$ of all the particles in the body together specify the instantaneous state of the rigid body. To understand its dynamics, we examine its Lagrangian.

8.3 Kinetic energy and inertia tensor or matrix

• The Lagrangian of the rigid body is $L = T - U$ where U is the external potential that the body moves in. Let us express the kinetic energy of the rigid body in terms of the center of mass velocity and angular velocity. The kinetic energy is just a sum of free particle KE of each constituent mass point labelled a, whose velocity vector we have denoted $\not{v}_a$ relative to the origin of the lab frame. So

$$T = \sum \frac{1}{2} m_a \not{v}_a^2 \equiv \sum \frac{1}{2} m \not{v}^2 \tag{215}$$

where the sum is over mass points. We will often suppress the index a that labels the points. Note that this index appears on $\mathbf{r}$, $\not{v}$ but not on $\mathbf{\Omega}$ or $\mathbf{V}$, which are properties of the body as a whole. Thus,

$$\begin{aligned} T &= \sum \frac{1}{2} m \left(\mathbf{V} + \mathbf{\Omega} \times \mathbf{r}\right)^2 = \sum \frac{1}{2} m \mathbf{V}^2 + \sum m \mathbf{V} \cdot \mathbf{\Omega} \times \mathbf{r} + \sum \frac{1}{2} m (\mathbf{\Omega} \times \mathbf{r})^2 \\ &= \frac{1}{2} M \mathbf{V}^2 + (\mathbf{V} \times \mathbf{\Omega}) \cdot \sum m \mathbf{r} + \frac{1}{2} \sum m \left[\Omega^2 r^2 - (\mathbf{\Omega} \cdot \mathbf{r})^2\right] \end{aligned} \tag{216}$$

We simplified the second term using the cyclic symmetry of the formula for the volume of a parallelepiped $\mathbf{V} \cdot \mathbf{\Omega} \times \mathbf{r} = \mathbf{r} \cdot \mathbf{V} \times \mathbf{\Omega}$. But the second term vanishes since the center of mass lies at the origin of the moving system: $\sum m\mathbf{r} = 0$. Ω, r are the magnitudes of $\mathbf{\Omega}, \mathbf{r}$. Thus the kinetic energy may be written as a sum of the translational kinetic energy of a body of mass M located at the center of mass and the kinetic energy of rotation about the center of mass

$$T = \frac{1}{2} M \mathbf{V}^2 + \frac{1}{2} \Omega_i \Omega_j \sum m \left(r^2 \delta_{ij} - r_i r_j\right) = \frac{1}{2} M \mathbf{V}^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j. \tag{217}$$

The rotational kinetic energy involves a 3×3 matrix called the inertia matrix/tensor (distinct from the identity matrix!)

$$I_{ij} = \sum_a m_a \left(r_a^2 \delta_{ij} - (r_a)_i (r_a)_j\right) = \iiint \rho(\mathbf{r}) \, \left(r^2 \delta_{ij} - r_i r_j\right) \, d^3\mathbf{r}. \tag{218}$$

We have written the formula for a rigid body with continuous mass distribution as well. The components of the inertia tensor are independent of time if $(r_a)_i$ are the components with respect to the co-moving frame. If instead, we use the components of $\mathbf{r}_a$ with respect to the lab frame, then I_{ij} will be time-dependent in general, this is often less convenient. The matrix I_{ij} is an intrinsic property of the mass distribution of the rigid body, the chosen origin and axes of the co-moving frame.

• The inertia matrix is real and symmetric matrix $I_{ij} = I_{ji}$. It is a positive matrix in the sense that the associated quadratic form ('rotational kinetic energy') is manifestly non-negative

$$T_{rot} = \frac{1}{2} I_{ij}\Omega_i\Omega_j = \frac{1}{2}\Omega^t I\Omega = \sum_a \frac{1}{2} m_a(\mathbf{\Omega}\times\mathbf{r}_a)^2 \geq 0. \tag{219}$$

We may write out the components of the inertia matrix,

$$I = \sum \begin{pmatrix} m(y^2+z^2) & -mxy & -mxz \\ -myx & m(z^2+x^2) & -myz \\ -mzx & -mzy & m(x^2+y^2) \end{pmatrix} \tag{220}$$

it is evidently the sum of the inertia matrices of each mass point in the body. The diagonal entries $I_{11} = \sum m(y^2+z^2), I_{22}, I_{33}$ are called the moments of inertia about the first, second and third axes of the rotating frame. In general, given any axis $\hat{n}$, the moment of inertia about $\hat{n}$ is defined as $I_{\hat{n}} = \sum_a m_a\rho_a^2$ where ρ_a is the perpendicular distance of point a from the axis. The parallel axis theorem relates the moment of inertia about an axis through the center of mass to the moment of inertia about a parallel axis $\hat{m}$: $I_{\hat{m}} = I_{\rm cm} + Md^2$, where d is the distance between axes and M is the total mass of the body.

• Being a real symmetric matrix, the inertia matrix may be diagonalized by an orthogonal transformation S that rotates the axes of the co-moving frame: $S^{-1}IS = D$ where D is the diagonal matrix of eigenvalues. As the matrix is positive, the eigenvalues are non-negative, they are called the principal moments of inertia, which we may order as $0 \leq I_1 \leq I_2 \leq I_3$. The eigenvectors of the inertia matrix may be chosen orthonormal and are called the principal axes of inertia. If the axes of the moving frame are chosen along the principal axes of inertia, then the inertia matrix is diagonal[12]

$$D = \begin{pmatrix} \sum_a m_a(y'^2_a + z'^2_a) & 0 & 0 \\ 0 & \sum_a m_a(z'^2_a + x'^2_a) & 0 \\ 0 & 0 & \sum_a m_a(x'^2_a + y'^2_a) \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{221}$$

Here x'_a, y'_a, z'_a are the components of the radius vector of a point in the body with respect to the principal axis basis. Note that the off diagonal entries vanish due to cancellations $\sum_a m_a x'_a y'_a = 0$ even though $m_a x'_a y'_a$ is, in general, non-zero for various particles in the body. In the principal axis basis, the rotational kinetic energy is particularly simple

$$T_{\rm rot} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right). \tag{222}$$

If all three principal moments of inertia are unequal, we call it an anisotropic rigid body. If one pair coincide, it is called a symmetrical top. For e.g. if $I_1 = I_2$ then the corresponding two principal axes may be chosen to be any pair of mutually perpendicular unit vectors in the corresponding x-y eigenplane. If all three eigenvalues coincide it is called a spherical top and the principal axes of inertia can be chosen as any orthonormal frame.

• If the body is concentrated along a straight line, say the z-axis, then it has no rotational inertia when spinning about the z-axis. Such a body is called a rigid rotator. Examples include

[12]The principal moments of inertia satisfy a triangle inequality $I_i + I_j \geq I_k$ where i, j, k are distinct indices from $1, 2, 3$.

a very thin pencil/dumbell or wire. Argue that the configuration space of such a rigid body is $\mathbb{R}^3 \times S^2$. It has only two rotational degrees of freedom which may be parametrized by points on a two-sphere which specify the direction in which the body is pointing. Note that the mass distribution need not be uniform along the z-axis. Since $x = y = 0$ for all particles, the centre of mass lies on the z-axis and we must have $I_3 = 0$ and $I_1 = I_2 = \sum_a m_a z_a^2$. Note that two of the triangle inequalities are saturated $I_1 + I_3 = I_2$ and $I_2 + I_3 = I_1$. The principal axes of inertia point along the z-axis and any pair of mutually orthogonal directions in the x-y plane.

• Consider a rigid body that is concentrated on a plane (say the x-y plane), like a flat plate or sheet of cardboard. $z = 0$ for all points on the body so the centre of mass lies on the x-y plane. It is clear from the explicit matrix representation of I that it is block diagonal and that the z-axis is one of the principal axes. The other two lie in the x-y plane. Let us choose the x and y axes to point along these principal axes of inertia. The principal moments of inertia are $I_1 = \sum my^2$, $I_2 = \sum mx^2$ and $I_3 = \sum m(x^2 + y^2)$. Notice that the triangle inequality is saturated $I_1 + I_2 = I_3$, this relation is called the perpendicular axis theorem.

8.4 Angular momentum of a rigid body

• The angular momentum of a system of particles is defined with respect to an origin. If we use the origin of the lab frame, then the radius vector of point labelled a in the rigid body is $\not{r}_a = \mathbf{R} + \mathbf{r}_a$ where $\mathbf{R}$ is the location of the centre of mass. If $\not{p}_a$ is the lab-frame momentum of the same particle, we must have $\not{p}_a = m_a \not{v}_a = m_a \mathbf{V} + m_a \boldsymbol{\Omega} \times \mathbf{r}_a$. Then the 'total' angular momentum about the origin of the lab frame is

$$\mathbf{L}_{\text{tot}} = \sum_a \not{r}_a \times \not{p}_a = \sum_a \mathbf{R} \times \not{p}_a + \sum_a m_a \vec{r}_a \times \mathbf{V} + \sum_a m_a \mathbf{r}_a \times (\boldsymbol{\Omega} \times \mathbf{r}_a) \tag{223}$$

Furthermore let $\mathbf{P} = \sum_a \not{p}_a$ be the total momentum ('centre of mass momentum') of the body, then

$$\mathbf{P} = \sum_a \not{p}_a = \sum_a m_a \not{v}_a = \sum_a m_a \mathbf{V} + \boldsymbol{\Omega} \times \sum_a m_a \mathbf{r}_a = M\mathbf{V} \tag{224}$$

by the definition of center of mass. Thus the total angular momentum is

$$\mathbf{L}_{\text{tot}} = \mathbf{R} \times \mathbf{P} + \left(\sum_a m_a \mathbf{r}_a \right) \times \mathbf{V} + \sum_a m_a \mathbf{r}_a \times (\boldsymbol{\Omega} \times \mathbf{r}_a) = \mathbf{L}_{\text{cm}} + \mathbf{L}_{\text{rot}} \tag{225}$$

The middle term is zero by the definition of center of mass. We see that the total angular momentum about the origin of lab frame splits into a centre of mass part $\mathbf{L}_{\text{cm}} = \mathbf{R} \times \mathbf{P}$ and a rotational part. We are primarily interested in the latter. $\mathbf{L}_{\text{rot}}$ is in fact the angular momentum about the center of mass (origin of co-moving frame). It is the angular momentum resulting from motion relative to the center of mass. $\mathbf{L}_{\text{rot}} = \sum_a \mathbf{r}_a \times \mathbf{p}_a$ where, $\mathbf{p}_a$ is the momentum relative to the center of mass, i.e., $\mathbf{p}_a = m_a \mathbf{v}_a = m_a(\boldsymbol{\Omega} \times \mathbf{r}_a)$. So we define

$$\begin{aligned} \mathbf{L} \;\equiv\; \mathbf{L}_{\text{rot}} &= \sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_a m_a \, \mathbf{r}_a \times (\boldsymbol{\Omega} \times \mathbf{r}_a) = \sum_a m_a (r_a^2 \boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \mathbf{r}_a)\, \mathbf{r}_a) \\ \Rightarrow \qquad L_i &= \sum m \left(r^2 \delta_{ij} - r_i r_j \right) \Omega_j = I_{ij} \Omega_j . \end{aligned} \tag{226}$$

Thus the 'rotational' angular momentum with respect to the centre of mass is related to the angular velocity via the inertia matrix. This is loosely analogous to how the translational

momentum of a point particle is related to its velocity via the mass $\mathbf{p} = m\mathbf{v}$. Momentum always points in the same direction as velocity. But in general, angular momentum points in the same direction as angular velocity only for an isotropic rigid body, for which the inertia matrix is a multiple of the identity. For a non-isotropic rigid body, the angular momentum points in the same direction as angular velocity only if a principal axis of inertia can be taken to point along the angular velocity vector.

• We note that $\mathbf{L}_{\rm rot}$ defined above is the angular momentum as defined by an inertial observer stationed at the center of mass. It is different from an angular momentum about the center of mass defined by an observer who moves and rotates with the body. For such an observer, the momenta of all particles in the body are zero, since nothing in the body moves relative to such an observer. Such an angular momentum is identically zero and not a useful concept for us.

8.5 Equations of motion of a rigid body

• The eom of the rigid body made of N point masses can be written using Newton's second law

$$m_a \ddot{\boldsymbol{\ell}}_a = \mathbf{f}_a \quad \text{for} \quad a = 1, \ldots, N. \tag{227}$$

$\mathbf{f}_a$ is the force acting on the $a^{\rm th}$ particle, including both external forces and (internal) forces due to other particles in the rigid body. These are $3N$ second order equations. However, a generic rigid body only has 6 degrees of freedom, irrespective of how large N may be. So the above equations are somewhat redundant. They do not make manifest the fact that the body is rigid (this is encoded in a complicated way in the internal parts of the forces $\mathbf{f}_a$). We seek a more global formulation of the eom, without reference to individual mass points. The configuration of the rigid body may be specified by giving the location $\mathbf{R}$ of the center of mass and a rotation $\vec{\phi}$ about the CM that brings the body to the desired orientation, relative to a reference orientation.

• The equations of motion for the rigid body may be formulated as equations for the centre of mass momentum $\mathbf{P}$ and for the angular momentum *about the centre of mass* $\mathbf{L}$. We will obtain these equations in the lab frame or any frame inertially related to it, the equations have the same form in all such frames by Galilean invariance. Note that the lab frame and the co-rotating frame are not related by galilean transformations, the co-rotating frame is in general a non-inertial frame. So the eom will take different forms in these two frames. We will transform the eom to the co-rotating frame in the next section.

• Let us work in the fixed lab frame. If $\mathbf{f}_a$ is the force on the $a^{\rm th}$ particle, then $\dot{\boldsymbol{\ell}}_a = \mathbf{f}_a$ is Newton's equation. Adding these up for all the particles we get $\dot{\mathbf{P}} = \mathbf{F}$ where $\mathbf{P} = \sum_a \boldsymbol{\ell}_a = M\mathbf{V}$ is the total momentum and $\mathbf{F} = \sum \mathbf{f}_a$ is the total force acting on the body. Here we need only include the external forces acting on the particles since the inter-particle forces balance out and cancel. $\dot{\mathbf{P}} = \mathbf{F}$ is the equation of motion in the lab frame. By Galilean invariance, it also takes the same form in any frame that is inertially related (via a translation/rotation/boost or some combination of these) to the lab frame.

• This equation of motion may also be obtained from the Lagrangian[13]

$$L = \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}I_{ij}\Omega_i\Omega_j - U(\mathbf{R},\phi) \quad \text{where} \quad \mathbf{V} = \dot{\mathbf{R}}. \tag{228}$$

• The corresponding equations of motion for the coordinate $\mathbf{R}$ are seen to reproduce what we got above

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{R}}} = \frac{\partial L}{\partial \mathbf{R}} \quad \Rightarrow \quad M\dot{\mathbf{V}} \equiv \dot{\mathbf{P}} \;=\; -\frac{\partial U}{\partial \mathbf{R}} \;=\; \mathbf{F} = \text{ force.} \tag{229}$$

Here we used the fact that the change in potential energy under a translation of the center of mass by $\delta\mathbf{R}$ is $\delta U = \sum_a \frac{\partial U}{\partial \boldsymbol{\imath}_a} \cdot \delta\mathbf{R}$ since all particles are translated by the same amount, $\delta\, \boldsymbol{\imath}_a = \delta\mathbf{R}$. So $\delta U = -\sum_a \mathbf{f}_a \cdot \delta\mathbf{R} = -\mathbf{F}\cdot\delta\mathbf{R}$.

• Next we compute the time derivative of the angular momentum about the centre of mass $\mathbf{L} = \sum_{\mathbf{a}} \mathbf{r_a} \times \mathbf{p_a}$ where $\mathbf{r}$ is the radius vector of the a^{th} particle and $\mathbf{p}$ its momentum relative to the center of mass. We want the equation of motion for $\mathbf{L}$ in the lab frame, not the co-rotating frame. But the equation of motion will take the same form in any frame that is related to the lab frame by a Galilean transformation. So for convenience, let us work in an inertial frame that at the instant considered is moving with velocity $\mathbf{V}$ with respect to the lab frame and has origin at the center of mass. In this frame, the center of mass is instantaneously at rest and the mass point a has radius vector $\mathbf{r}_a$, velocity $\mathbf{v}_a = \dot{\mathbf{r}}_a = \boldsymbol{\Omega} \times \mathbf{r}_a$ and momentum $\mathbf{p}_a = m\dot{\mathbf{r}}_a$. Moreover, $\dot{\mathbf{p}}_a = \mathbf{f}_a$ is the force on this particle

$$\text{So} \qquad \dot{\mathbf{L}} = \sum_a \left(\dot{\mathbf{r}}_a \times \mathbf{p}_a + \mathbf{r}_a \times \dot{\mathbf{p}}_a\right). \tag{230}$$

The first term vanishes as $\dot{\mathbf{r}}_a \times m\dot{\mathbf{r}}_a = 0$. In the second term, $\mathbf{r}_a \times \dot{\mathbf{p}}_a = \mathbf{r}_a \times \mathbf{f}_a = \mathbf{k}_a$ is the torque on the a^{th} particle about the centre of mass and so the second term is $\mathbf{K} = \sum_a \mathbf{k}_a$ which is the total torque on the body about the centre of mass. Thus, the equation for evolution of the angular momentum of the rigid body is $\dot{\mathbf{L}} = \mathbf{K}$. Here both the angular momentum and torque are defined with respect to the centre of mass of the rigid body and this equation is written in an inertial frame moving at velocity $\mathbf{V}$ relative to the lab frame. Moreover, $\mathbf{L} = \mathbf{I}\,\boldsymbol{\Omega}$, so we could also say $\mathbf{I}\,\dot{\boldsymbol{\Omega}} = \mathbf{K}$. In the absence of any external torque about the centre of mass (e.g. if there are no external forces), the angular momentum of the rigid body about its CM is independent of time $\dot{\mathbf{L}} = 0$. This equation of motion takes the same form in the lab frame, by Galilean invariance.

• The equation for $\dot{\mathbf{L}}$ also follows from the above Lagrangian. We think of the components of angular velocity $\boldsymbol{\Omega}$ as the rate of change $\dot{\phi}$ of angular variables $\vec{\phi}$ specifying the orientation of the rigid body, for instance, via an axis and an angle to specify a rotation from a reference orientation. We will be more explicit about these angular variables later on. Then the LHS of Lagrange's equations for the angular variables is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{\phi}}} = \frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\Omega}} = \frac{d}{dt}(I\boldsymbol{\Omega}) = \dot{\mathbf{L}}. \tag{231}$$

As for the RHS, let us show that $\frac{\partial L}{\partial \phi} = \mathbf{K}$. Now $\frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi}$ is (minus) the change in potential energy due to an infinitesimal rotation $\delta\phi$. The change in potential energy δU due to an

[13]The external potential energy U could depend both on the location $\mathbf{R}$ of the center of mass as well as the orientation of the body, specified, say, by angular variables ϕ which could specify an axis and an angle that determine a rotation that would bring the body to a reference orientation.

infinitesimal rotation $\delta\phi$ (about an axis passing through the CM) is minus the work done by the external forces. Under a rotation about the CM, the CM $\mathbf{R}$ does not move, so $\delta\,\mathbf{r}_a = \delta\mathbf{r}_a$, so the change in potential energy is

$$\delta U = -\sum_a \mathbf{f}_a\cdot\,\delta\,\mathbf{r}_a = -\sum_a \mathbf{f}_a\cdot(\delta\phi\times\mathbf{r}_a) = -\delta\phi\cdot\sum_a \mathbf{r}_a\times\mathbf{f}_a = -\delta\phi\cdot\mathbf{K} \;\Rightarrow \lim_{\delta\phi\to 0}\frac{\delta U}{\delta\phi} = -\mathbf{K}. \quad (232)$$

We used the cyclic symmetry of the scalar triple product. So the Euler-Lagrange equations imply the law of evolution of angular momentum (relative to the CM) of the rigid body

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = \frac{\partial L}{\partial\phi} \;\Rightarrow\; \dot{\mathbf{L}} \;=\; -\frac{\partial U}{\partial\phi} \;=\; \mathbf{K} = \text{ total torque about CM.} \quad (233)$$

8.6 Force-free motion of rigid bodies

• Now consider a rigid body in the absence of any external forces. If the centre of mass was initially at rest, it will remain so. More generally, the center of mass will move along a straight line since $\ddot{\mathbf{R}} = 0$. Let us choose our inertial frame to be such that its origin always lies at the CM, i.e., the inertial frame moves at constant velocity $\mathbf{V}$ relative to the fixed lab frame. So we have ensured that the CM is at rest in the inertial frame. Of course, the body could rotate while the centre of mass remains at rest. Since there are no external forces, there are no external torques either, about any point. So the angular momentum about any point must be conserved. In particular, the angular momentum about the CM must be constant in time, provided it is measured with respect to an inertial system. It is important to realize that the components of the angular momentum vector, with respect to the co-moving frame, are in general *not* constant in time, since the frame is rotating relative to the inertial frame. We will see examples of this.

• Let us illustrate some consequences of conservation of angular momentum and the formula $\mathbf{L} = I\boldsymbol{\Omega}$, for force-free motion of simple rigid bodies. For a spherical top the principal moments of inertia are all equal. I is a multiple of the identity in any basis, We may write $\mathbf{L} = I_1\boldsymbol{\Omega}$, so the angular velocity is just a multiple of the angular momentum, both point in the same direction and are both constant in time. In particular, force free motion of a spherical top consists of uniform rotation about some axis that is fixed in the lab frame. The direction of this axis and the rate of rotation $|\Omega|$ are determined by initial conditions. The conserved energy $E = \frac{1}{2}I_1(\Omega_1^2+\Omega_2^2+\Omega_3^2) = \frac{1}{2}I_1|\boldsymbol{\Omega}|^2 = \frac{|\mathbf{L}|^2}{2I_1}$ is also determined by initial conditions.

8.6.1 Free motion of rigid rotator

• Next consider a rigid rotator (collinear rigid body). Suppose the third principal axis points along the axis of the rotator. In this case the principal moments of inertia are $I_1 = I_2$ and $I_3 = 0$. Since all the masses lie along the axis of the rotator, in the formula for angular momentum $\vec{L} = \sum_a \vec{r}_a\times\vec{p}_a$, $\vec{r}_a$ is along the axis of the rotator. So $\vec{L}$ must be orthogonal to the axis of the rotator at all times. We could have reached this conclusion by a different argument as well. From $L = I\Omega$ written in the principal axis basis, we see that the component of angular momentum in the direction of the axis of the rotator must vanish ($L_3 = I_3\Omega_3 = 0$) irrespective of what Ω_3 is. So the angular momentum must always point in a direction orthogonal to the axis of the rotator. Since $\mathbf{L}$ is a constant vector, the axis of the rotator must always lie in the

plane orthogonal to $\vec{L}$. In other words, the rotator must rotate in a fixed plane with respect to the lab frame. Let us for simplicity call this the X-Y plane (in the lab system) so that $\mathbf{L}$ points along Z.

• So for an infinitesimal rotation, $\delta\vec{\phi} \propto \hat{Z}$. It follows that the angular velocity $\Omega = \lim_{\delta t \to 0} \frac{\delta\vec{\phi}}{\delta t}$ also points along $\hat{Z}$. So both $\mathbf{L}$ and $\mathbf{\Omega}$ point along $\hat{Z}$.

• If the z axis is chosen along the axis of the rod, $x = y = 0$ for all mass points and so the inertia tensor becomes diagonal $I = \mathrm{diag}(I_1, I_2, 0)$ irrespective of how we choose the x, y axes in the plane orthogonal to z. It follow that $L_1 = I_1\Omega_1$, $L_2 = I_1\Omega_2$ and $L_3 = 0$ where $1, 2, 3$ refer to components in principal axis frame. It follows that $\mathbf{L} = I_1\mathbf{\Omega}$ so $\mathbf{\Omega}$ is a constant vector in space, just like $\mathbf{L}$ is. Moreover, the rate at which the rod rotates (axis 'precesses' about $\mathbf{L}$) is independent of time and equal to $|\mathbf{\Omega}| = L/I_1$ where L is the magnitude of the angular momentum vector. So the most general free motion of a rigid rotator is uniform rotation in a plane fixed with respect to the lab frame and orthogonal to the direction of angular momentum.

• We may also specify a convenient body fixed frame which is also a principal axes frame. z was taken along the axis of the rotator ($I_3 = 0$). We may take x along the fixed Z axis (direction of $\mathbf{L}$), which is a direction that is fixed both in the body and in space. Then y must lie in the XY plane in such a way that xyz is a right-handed system. With these choices, we may write the angular momentum vector in terms of its components in both the lab and principal-axes-body-fixed frames.

$$\mathbf{L} = L\hat{Z} = L\hat{x} \quad \text{and} \quad \mathbf{\Omega} = \frac{L}{I_1}\hat{Z} = \frac{L}{I_1}\hat{x}. \tag{234}$$

8.6.2 Free motion of symmetrical top

• Force-free motion of a symmetric top is a combination of spinning on its axis and precession of the axis about a fixed direction in space, and at a fixed angle (there is no nutation in the absence of gravity).

• Conservation of angular momentum and the relation between angular momentum and angular velocity allow us to understand some aspects of the free motion of a symmetrical top as well. Following Landau and Lifshitz (see fig. 6), consider a symmetrical top with $0 < I_1 = I_2 \neq I_3 \neq 0$. The angular momentum in space is of course a constant vector $\mathbf{L}$. Let us choose the Z axis of the lab frame along $\mathbf{L}$. The axis of the top is along the third principal axis $\hat{x}_3 = \hat{z}$. The first two principal axes of inertia $\hat{x}_1, \hat{x}_2$ may be freely chosen (orthogonal to the z-axis and forming a right-handed system) since the corresponding eigenvalues are equal $I_1 = I_2$.

• However, not every such choice of principal axes are fixed in the body. A principal axes frame is not necessarily a body fixed frame or vice versa. However, in this section, we do not need to choose a co-moving frame. In fact, it is not possible, in general, to specify a co-moving frame relative to the lab frame, without prior knowledge of how the body moves! So we will simply choose a convenient set of principal axes with respect to which the inertia tensor is diagonal. This principal axis frame will not qualify as a co-moving frame.

• Let us choose the second principal axis of inertia $\hat{x}_2$ to be always perpendicular to the plane spanned by $\mathbf{L}$ and the axis of the top[14]. Then the $\hat{x}_1$ principal axis must lie in the same

[14]Here we exclude the possibility that the axis of the top, points along $\mathbf{L}$. This case ($\theta = 0$) will be treated

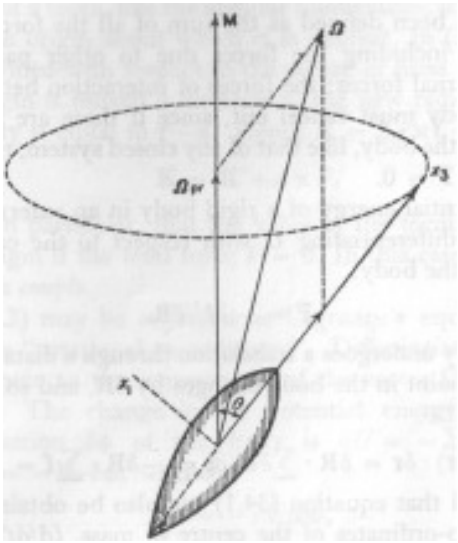


Figure 6: Fig 46 of symmetric top from Landau & Lifshitz. $\mathbf{M} = \mathbf{L}$ is vertical and axis is along $\hat{x}_3 = \hat{z}$. The x_1 principal axis has been marked, and chosen to lie in the plane of the axis of the top and the angular momentum vector. It is not fixed in the body, if the body spins on its axis.

plane as $\mathbf{L}$ and $\hat{x}_3$, in such a way that the x_1, x_2, x_3 axes form a right handed orthonormal system. Since the x_2 axis is always orthogonal to $\mathbf{L}$, it follows that the component of angular momentum along the x_2 axis, $L_2 = 0$. This implies $\Omega_2 = L_2/I_2 = 0$ as $I_2 \neq 0$. So $\mathbf{\Omega}$ always lies in the plane spanned by $\mathbf{L}$ and the axis of the top. It follows that the velocity of any point on the axis of the top, $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ always points perpendicular to that plane. So the axis of the top precesses about the direction of angular momentum in space, sweeping out a cone. Let us denote the angle between the axis of the top and $\mathbf{L}$ by the symbol θ. Let us argue that θ is time independent so that the cone has a fixed opening angle. $\mathbf{v}$ for a point on the axis of the top would have to have a component in the plane containing $\mathbf{L}$ and the axis of the top for the angle θ to change. However, as noted above, $\mathbf{v}$ (for points on the axis of the top) always points in a direction perpendicular to this plane. So θ must be fixed by the initial conditions. We will see below that the rate of precession is also constant in time. We will also show that the length of Ω is constant in time. In addition to precessing about the Z axis, the top also spins about its own axis. We wish to find the angular speeds of precession $\Omega_{\rm pr}$ and spin $\Omega_{\rm spin}$. Let us find expressions for these two angular velocities in terms of the angle θ and the constant magnitude of angular momentum $|\mathbf{L}| = L$ and the principal moments of inertia. It helps to expand $\mathbf{L}$ and $\mathbf{\Omega}$ in the principal axis basis. $\mathbf{\Omega} = \Omega_1\hat{x}_1 + \Omega_3\hat{x}_3$ and $\mathbf{L} = L_1\hat{x}_1 + L_3\hat{x}_3$ where $L_1 = L\sin\theta$ and $L_3 = L\cos\theta$ where θ is the angle between $\mathbf{L}$ and $\hat{z}$. By constancy of the magnitude L and θ, the components L_1, L_3 of $\mathbf{L}$ in the principal axis frame are independent of time, $\mathbf{L}$ is a constant vector both in the principal axis frame and in space. Moreover, $\Omega_1 = L\sin\theta/I_1$ and $\Omega_3 = L\cos\theta/I_3$ are also constant in time. So $\mathbf{\Omega}$ is a constant vector in the principal axis frame. But $\mathbf{\Omega}$ is not a constant vector in space, the plane in which it lies goes round and round the Z axis as seen in the figure.

• The rate at which the top spins on its axis may be obtained by decomposing $\mathbf{\Omega}$ into two

separately.

parts $\mathbf{\Omega} = \Omega_3\hat{z} + \Omega_1\hat{x}$, the second term does not cause any spinning motion, it simply moves the axis of the top as a whole. So $\Omega_{\text{spin}} = \Omega_3 = \mathbf{\Omega}\cdot\hat{x}_3 = \frac{L_3}{I_3} = \frac{L\cos\theta}{I_3}$. We argued above that θ is independent of time, so the top spins at a constant rate on its axis. Moreover, a symmetric top *must* spin on its axis, as long as the axis is inclined at anything other than right angles to the direction of $\mathbf{L}$. θ is fixed by initial conditions.

• On the other hand, to find the precession rate of the axis of the top about the $\hat{Z}$ axis, we will decompose $\mathbf{\Omega}$ as a sum of two vectors, one of which does not cause any precession and thereby identify the rate of precession. We write $\mathbf{\Omega}$ as a (non-orthogonal in general!) linear combination of $\hat{z}$ and $\hat{Z}$, $\vec{\Omega} = \Omega_{(3)}\hat{x}_3 + \Omega_{\text{pr}}\hat{Z}$. Here we assume $\theta \neq 0$, since otherwise $\hat{Z}, \hat{z}$ are not linearly independent. The first of these (which is not equal to $\Omega_3\hat{x}_3$ in general) does not produce any displacement of the axis of the top. The second component gives the precession rate. To find the precession rate Ω_{pr} we simply take the dot product with $\hat{x}_1$ and use the fact that $\hat{x}_1\cdot\hat{x}_3 = 0$:

$$\Omega_1 = \vec{\Omega}\cdot\hat{x}_1 = \Omega_{\text{pr}}\hat{Z}\cdot\hat{x}_1 = \Omega_{\text{pr}}\sin\theta \;\;\Rightarrow\;\; \Omega_{\text{pr}} = \frac{\Omega_1}{\sin\theta} = \frac{L_1}{I_1\sin\theta} = \frac{L}{I_1}. \tag{235}$$

So we see that the precession rate is constant and non-zero if $\mathbf{L}$ is not the zero vector (and $\theta \neq 0$). The axis of the top rotates uniformly about $\mathbf{L}$ in space. Since $\mathbf{\Omega}$ lies in the same plane as $\mathbf{L}$ and the axis of the top, it follows that $\mathbf{\Omega}$ also precesses about $\mathbf{L}$ at the same rate $\Omega_{\text{pr}} = L/I_1$. Interestingly, the rate of precession is independent of opening angle θ.

• So far we excluded the case $\theta = 0$. Suppose the axis of the top (the 3rd principal axis $\hat{z}$) always points along the fixed direction of angular momentum in space $\mathbf{L} = L\hat{Z}$. Then $\mathbf{L} = L\hat{z}$. So the components of L in any principal axis basis are $L_1 = L_2 = 0, L_3 = L$. It follows that $\Omega_1 = L_1/I_1 = 0$ and $\Omega_2 = L_2/I_2 = 0$, so $\mathbf{\Omega} = \Omega_3\hat{z} = (L/I_3)\hat{Z}$. So like $\mathbf{L}$, $\mathbf{\Omega}$ too is a fixed vector in space. Its magnitude $\Omega = L/I_3$ is the rate Ω_{spin} at which the top spins on its axis. Since the axis always points in the same direction, it does not precess, $\Omega_{\text{pr}} = 0$. We may obtain the case $\theta = 0$ as a limit of the previous analysis. For $\theta > 0$ we found $\Omega_{\text{spin}} = L\cos\theta/I_3$ which reduces to L/I_3 in the limit $\theta \to 0$. To find the limiting value of the precession rate $\Omega_{\text{pr}} = \frac{L_1}{I_1\sin\theta}$, we must bear in mind that as $\theta \to 0$, $L_1 \to 0$ as well. The physical process of taking the limit must be such that L_1 vanishes faster than $\sin\theta$, so that $\Omega_{\text{pr}} \to 0$ as $\theta \to 0$.

8.7 Euler angles

• The purely geometric/pictorial approach used above has its limitations if we wish to study the motion of an anisotropic top or one that is subject to external forces. It helps to have a coordinate system on configuration space. This will allow us to write down the differential equations of motion and look for solutions.

• Euler angles are a way of parametrizing the rotational degrees of freedom of a rigid body. They give us a way of specifying the orientation of the co-rotating frame $x = x_1, y = x_2, z = x_3$ with respect to the inertial frame X, Y, Z, both of which are right-handed systems. Since we are interested in the relative orientation, we may, without loss of generality, assume that the two frames have a common origin (say the center of mass). At any instant, the co-rotating frame is related to the fixed frame by a rotation. So Euler angles parametrize points on the rotation group $\mathrm{SO}(3)$. Now, the XY and xy planes intersect along a line ON which is called the line of nodes. This line is of course orthogonal to both the $\hat{Z}$ and $\hat{z}$ axes, and the direction of ON is chosen along $\hat{Z}\times\hat{z}$. Now the orientation of the xyz axes relative to the fixed XYZ axes

is specified as follows. θ is the angle between the Z and z axes. The line of nodes ON on the XY plane is fixed by saying that it makes an angle ϕ with the X axis. This fixes the xy plane as it must be perpendicular to z and contain the line of nodes. Finally, the x axis is fixed by saying it makes an angle ψ with the line of nodes. Note that there are other conventions for specifying the Euler angles. The Euler angles θ, ϕ, ψ are generalised coordinates that fix the angular orientation of a rigid body. The corresponding generalised velocities are their time derivatives $\dot{\theta}, \dot{\phi}, \dot{\psi}$.

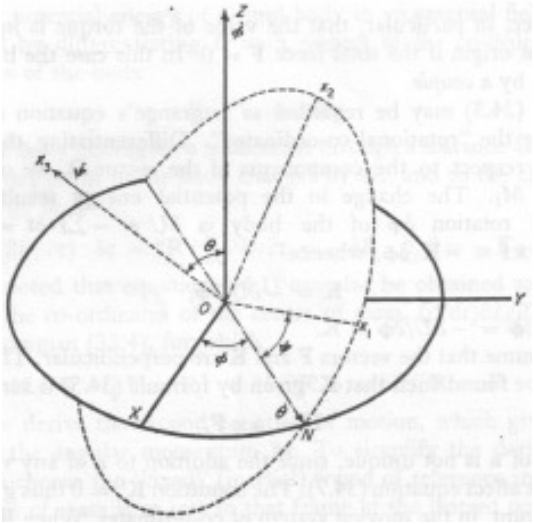


Figure 7: Euler angles and their time derivatives, from Landau and Lifshitz, Mechanics (fig. 47).

• The body fixed frame $\hat{x}, \hat{y}, \hat{z}$ may be obtained from the lab frame $\hat{X}, \hat{Y}, \hat{Z}$ by a sequence of three rotations by the Euler angles about suitably chosen axes. Suppose the body-fixed frame initially coincides with the lab frame (and has the same origin). Then we first rotate the body frame by an angle ϕ counter clockwise about the Z axis. As a result, the rotated $\hat{x}$ will now point along the *intended* line of nodes while $\hat{z}$ continues to coincide with $\hat{Z}$. Next we rotate the body frame by an angle θ counter clockwise about the new $\hat{x}$ axis (line of nodes). As a result of this, the new $\hat{z} = \hat{x}_3$ axis has reached its desired orientation. Finally, we rotate the body frame by an angle ψ counter-clockwise about the new $\hat{z}$ axis. As a result, $\hat{x}$ moves off the XY plane and makes an angle ψ with the line of nodes, and reaches its desired orientation.

• Now consider an infinitesimal rotation of the body (and co-moving frame) that is made up of small increments $\delta\theta, \delta\phi, \delta\psi$ in a small time δt. $\delta\vec{\theta}$ denotes an infinitesimal rotation by angle $\delta\theta$ about a certain axis. From the figure, we see that a small change in θ holding ϕ, ψ fixed is a rotation about the line of nodes. So we say that $\delta\vec{\theta}$ points along the line of nodes ON. Letting $\delta t \to 0$, $\dot{\vec{\theta}} = \dot{\theta}\,\hat{\mathrm{ON}}$. Similarly, $\dot{\vec{\phi}}$ points along $\hat{Z}$ so $\dot{\vec{\phi}} = \dot{\phi}\hat{Z}$ and $\dot{\vec{\psi}} = \dot{\psi}\hat{z}$. We add up the effects of these three small rotations per unit time to get the angular velocity vector $\vec{\Omega} = \dot{\vec{\theta}} + \dot{\vec{\phi}} + \dot{\vec{\psi}}$. Its components with respect to the co-rotating frame xyz are denoted $\Omega_1, \Omega_2, \Omega_3$. We wish to express the components Ω_i of angular velocity in terms of the generalised velocities $\dot{\theta}, \dot{\phi}, \dot{\psi}$.

• Let us denote by $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ the components of the vector $\dot{\vec{\theta}}$ in the three directions of the co-rotating xyz frame and similarly for $\dot{\phi}_i$ and $\dot{\psi}_i$; i.e., $\theta_1 = \dot{\vec{\theta}} \cdot \hat{x}$ etc. Then by some trigonometry

we determine the components.

$$\begin{aligned}\dot\theta_1 &= \dot\theta\cos\psi,\ \ \dot\theta_2 = -\dot\theta\sin\psi,\ \ \text{and}\ \ \dot\theta_3 = 0;\\ \dot\psi_1 &= \dot\psi_2 = 0,\ \ \text{and}\ \ \dot\psi_3 = \dot\psi;\\ \dot\phi_1 &= \dot\phi\sin\theta\sin\psi,\ \ \dot\phi_2 = \dot\phi\sin\theta\cos\psi,\ \ \dot\phi_3 = \dot\phi\cos\theta\end{aligned} \tag{236}$$

In the last equation, we used the fact that $\hat Z$ is $\perp$ to the line of nodes. So the projection of $\vec{\dot\phi} \propto \hat Z$ onto the xy plane must also be perpendicular to the line of nodes. It follows that this projection makes an angle ψ with the y axis. Combing these, we get $\Omega_1 = \vec\Omega\cdot\hat x = (\vec{\dot\theta}+\vec{\dot\phi}+\vec{\dot\psi})\cdot\hat x = \dot\theta_1+\dot\psi_1+\dot\phi_1$ etc:

$$\Omega_1 = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi,\quad \Omega_2 = -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi,\quad \Omega_3 = \dot\psi + \dot\phi\cos\theta. \tag{237}$$

Now, if x, y, z are taken to point along the principal axes of inertia, the formula for rotational kinetic energy simplifies to $T = \frac12\sum_j I_j\Omega_j^2$, where I_j are the principal moments of inertia. We may substitute for $\Omega_{1,2,3}$ from above to express the kinetic energy in terms of the Euler angles and their time derivatives:

$$\begin{aligned}T &= \frac{I_1}{2}\left(\dot\theta^2\cos^2\psi + \dot\phi^2\sin^2\theta\sin^2\psi + 2\dot\theta\dot\phi\sin\theta\sin\psi\cos\psi\right)\\ &+ \frac{I_2}{2}\left(\dot\phi^2\sin^2\theta\cos^2\psi + \dot\theta^2\sin^2\psi - 2\dot\theta\dot\phi\sin\theta\cos\psi\sin\psi\right) + \frac{I_3}{2}\left(\dot\psi^2 + \dot\phi^2\cos^2\theta + 2\dot\psi\dot\phi\cos\theta\right)\end{aligned} \tag{238}$$

• For a symmetric top, with $I_1 = I_2 \neq I_3$ this simplifies

$$\begin{aligned}T_{\text{symm top}} &= \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2\\ &= \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi^2 + \dot\phi^2\cos^2\theta + 2\dot\psi\dot\phi\cos\theta\right).\end{aligned} \tag{239}$$

For a spherical top $(I_1 = I_2 = I_3)$, it further simplifies

$$T_{\text{sph top}} = \frac12 I_1(\dot\theta^2 + \dot\phi^2 + \dot\psi^2 + 2\dot\psi\dot\phi\cos\theta). \tag{240}$$

• We could also obtain this formula by a judicious choice principal axes. Let us choose the z axis along the third principal axis of inertia corresponding to I_3. For a symmetrical top $I_1 = I_2$. Now we use the freedom of choosing the first two principal axes as any pair of orthogonal vectors perpendicular to z. Let us choose (at the instant considered), the first principal axis $x_1 = x$ to point along the line of nodes ON. Then $\psi = 0$ at the instant considered (this does not mean $\dot\psi$ is zero!). x_2 is determined by the need for x_1, x_2, x_3 to be a right-handed system. So

$$\Omega_1 = \dot\theta,\quad \Omega_2 = \dot\phi\sin\theta,\quad \Omega_3 = \dot\psi + \dot\phi\cos\theta. \tag{241}$$

Squaring and adding $\frac12 I_1\Omega_1^2 + \frac12 I_2\Omega_2^2 + \frac12 I_3\Omega_3^2$ we recover the above expression T_{rot} for a symmetric top. Let us now use these formulae in terms of Euler angles to find the rates of precession and spin of a freely rotating symmetrical top. Suppose the constant angular momentum vector points along the Z-axis of the lab frame. Let us choose the z axis of the co-rotating frame to point along the axis of the top. Then the rate at which the top spins on its axis is Ω_3. And the rate at which the axis of the top ($\hat z$) precesses about the angular momentum vector is equal to the rate at which the line of nodes goes round the $\hat Z$ axis, namely $\dot\phi$. We wish to express these rates

in terms of L and θ. To introduce L we first note that the components of angular momentum along the co-rotating axes are

$$L_1 = I_1\Omega_1 = I_1\dot{\theta}, \quad L_2 = I_2\Omega_2 = I_2\dot{\phi}\sin\theta \quad \text{and} \quad L_3 = I_3\Omega_3 = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) \tag{242}$$

On the other hand, since $\mathbf{L}$ points along $\hat{Z}$, its components along the principal axes must be

$$L_1 = \mathbf{L}\cdot\hat{x} = 0, \quad L_2 = \mathbf{L}\cdot\hat{y} = L\sin\theta \quad \text{and} \quad L_3 = \mathbf{L}\cdot\hat{z} = L\cos\theta. \tag{243}$$

L_1 vanishes since x has been chosen along the line of nodes, which is perpendicular to the Z axis. Comparing the two formulae, we express the relations between angular velocities and angular momenta as relations between the Euler angles and their time-derivatives, and the magnitude of angular momentum

$$\dot{\theta} = 0, \qquad I_2\dot{\phi} = L \qquad \text{and} \qquad I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = L\cos\theta \qquad \text{or} \qquad \dot{\psi} = L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right). \tag{244}$$

These relations are enough to give us an expression for the rate at which the top spins on its axis

$$\text{rate of spin } = \mathbf{\Omega}\cdot\hat{z} = \Omega_3 = \frac{L_3}{I_3} = \frac{L}{I_3}\cos\theta. \tag{245}$$

Note that $\dot{\psi}$ is not the rate at which the top spins on its axis. We will identify the physical meaning of $\dot{\psi}$ in the next section. Meanwhile, we find the rate of precession, which turns out to be a constant:

$$\text{precession rate } = \Omega_{pr} = \dot{\phi} = \frac{L_2}{I_2\sin\theta} = \frac{L\sin\theta}{I_2\sin\theta} = \frac{L}{I_1}. \tag{246}$$

These formulae agree with what we obtained by other means in the last section. For instance, $0 = L_1 = I_1\dot{\theta}$ so $\dot{\theta} = 0$. So the axis of the symmetric top remains at a constant angle relative to $\vec{L}$.

8.8 Euler equations for a rigid body in body-fixed frame

• As we have seen above, the eom of a rigid body may be formulated as equations for the centre of mass momentum $\mathbf{P} = M\mathbf{V}$ and for the angular momentum $\mathbf{L} = \mathbf{I}\,\mathbf{\Omega}$ about the centre of mass.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{L}}{dt} = \mathbf{K} \tag{247}$$

where $\mathbf{F} = \sum_{\mathbf{a}}\mathbf{f_a}$ and $\mathbf{K} = \sum_{\mathbf{a}}\mathbf{r_a}\times\mathbf{f_a}$ are the external force and external torque about the centre of mass. Here, the rates of change of both P and L are measured with respect to an inertial observer (e.g. lab frame). Now we wish to write these equations for the time evolution of $\mathbf{P}$ and $\mathbf{L}$ in the (non-inertial) co-rotating frame. The rotation of the frame imparts a time dependence even to a vector that may be fixed in the lab frame, so we should expect the equations to look a bit different. In particular, the angular momentum, which is fixed in space is in general not a constant vector with respect to the co-moving frame.

• As we have seen in the case of the force free motion of a symmetrical top, even if the angular momentum is constant in the lab frame, the angular velocity vector (in space) need not be constant in time, indeed the top could precess about the constant angular momentum vector.

• Let $\mathbf{A}$ be a vector such as angular momentum or linear momentum of the rigid body or of a mass point. We wish to relate its time dependence with respect to the lab frame to that in a frame rotating with instantaneous angular velocity $\mathbf{\Omega}$. If the vector is fixed in the rotating frame ($\dot{\mathbf{A}}_{rot} = 0$) then its time dependence in the lab frame arises purely from the rotation and is given by $\dot{\mathbf{A}}_{lab} = \mathbf{\Omega} \times \mathbf{A}$. The reason is the same as the one we gave in deriving the second term of the equation $\mathbf{v} = \mathbf{V} + \mathbf{\Omega} \times \mathbf{r}$ at the beginning of our study of rigid bodies. More generally, the vector $\mathbf{A}$ may be changing with respect to the rotating frame as well. Combining these two,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{lab} = \left(\frac{d\mathbf{A}}{dt}\right)_{rot} + \mathbf{\Omega} \times \mathbf{A} \tag{248}$$

Thus in the body-fixed frame we have a system of six equations:

$$\dot{\mathbf{P}} + \mathbf{\Omega} \times \mathbf{P} = \mathbf{F} \quad \text{and} \quad \dot{\mathbf{L}} + \mathbf{\Omega} \times \mathbf{L} = \mathbf{K} \quad \text{where} \quad \mathbf{L} = \mathbf{I}\mathbf{\Omega}. \tag{249}$$

The dot denotes time derivative with respect to the co-rotating frame. The equations for evolution of angular momentum components in the body-fixed frame are (the last two equations are got by cyclic permutation of indices)

$$\dot{L}_1 + (\Omega_2 L_3 - \Omega_3 L_2) = K_1, \qquad \dot{L}_2 + (\Omega_3 L_1 - \Omega_1 L_3) = K_2, \qquad \dot{L}_3 + (\Omega_1 L_2 - \Omega_2 L_1) = K_3. \tag{250}$$

• Here the components L_i and Ω_j are all unknown, but they are related via the inertia tensor. The relation between angular momentum and angular velocity is simplest in a principal axis frame, where the inertia matrix is diagonal. To exploit this simplicity we choose the axes of the co-rotating frame to point along the principal axes of inertia so that $L_i = I_i \Omega_i$ for each $i = 1, 2, 3$. The resulting equations were derived by Euler and bear his name. Assuming none of the principal moments of inertia vanish and defining $a_{ij} = I_i^{-1} - I_j^{-1}$,

$$\begin{aligned} &\dot{L}_1 + \left(\frac{1}{I_2} - \frac{1}{I_3}\right) L_2 L_3 = K_1, \qquad \dot{L}_2 + \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L_3 L_1 = K_2 \quad \text{and} \quad \dot{L}_3 + \left(\frac{1}{I_1} - \frac{1}{I_2}\right) L_1 L_2 = K_3 \\ &\text{or} \quad \dot{L}_1 + a_{23} L_2 L_3 = K_1, \qquad \dot{L}_2 + a_{31} L_3 L_1 = K_2, \quad \text{and} \quad \dot{L}_3 + a_{12} L_1 L_2 = K_3. \end{aligned} \tag{251}$$

It is also of interest to find the time evolution of the components of angular velocity with respect to the body fixed principal axes. Rather than try to extend the above formula to the case $\mathbf{A} = \mathbf{\Omega}$, we simply write $L_i = I_i \Omega_i$ in the Euler equations and obtain

$$\dot{\Omega}_1 + \left(\frac{I_3 - I_2}{I_1}\right) \Omega_2 \Omega_3 = K_1, \quad \dot{\Omega}_2 + \left(\frac{I_1 - I_3}{I_2}\right) \Omega_3 \Omega_1 = K_2 \text{ and } \dot{\Omega}_3 + \left(\frac{I_2 - I_1}{I_3}\right) \Omega_1 \Omega_2 = K_3. \tag{252}$$

• The Euler equations could be written as second order ODEs for the Euler angles by substituting for Ω_i in terms of θ, ϕ, ψ using our formulae from the previous section. But we could also regard them as first order equations specifying the evolution of the components of the angular momentum vector. Once the time-evolution of L_i or Ω_i have been found, we would then find the time dependence of the Euler angles by solving the 1st order equations for the Euler angles using Ω_i as input. Knowledge of Euler angles as a function of time would give us the orientation of the rigid body relative to the lab frame at all times.

• Note that the torque on the RHS of the Euler equations, $\mathbf{K} = \sum_a \mathbf{r_a} \times \mathbf{f}_a$ depends on the instantaneous location and orientation of the body (which depends for instance on $\mathbf{R}$ which must be determined from $\dot{\mathbf{R}} = \mathbf{P}$ by solving the momentum equations $\dot{\mathbf{P}} + \mathbf{\Omega} \times \mathbf{P} = \mathbf{F}$. Thus the

Euler equations in general are a complicated system that couple the rotational and translational motion. However, they simplify in the absence of external forces, in which case the equations for $\mathbf{L}$ decouple from those for $\mathbf{P}$. So for force free motion, the three Euler equations for L_1, L_2, L_3 are a self-contained system of quadratically non-linear ordinary differential equations for the rotational dynamics in the co-rotating principal axis frame.

8.8.1 Euler equations for force-free motion of symmetric top

• Let us consider the Euler equations in the case of the free motion of a symmetric top, for which $0 < I_1 = I_2$. So the axis of the top is the third principal axis, i.e., the z-axis of the co-moving frame. Euler's equations become

$$\dot{L}_1 + aL_2L_3 = 0, \quad \dot{L}_2 - aL_3L_1 = 0 \quad \text{and} \quad \dot{L}_3 = 0 \quad \text{where} \quad a = \frac{1}{I_2} - \frac{1}{I_3} = \frac{1}{I_1} - \frac{1}{I_3}. \tag{253}$$

So L_3 is a constant while L_1, L_2 obey the coupled equations

$$\dot{L}_1 + \omega L_2 = 0 \quad \text{and} \quad \dot{L}_2 - \omega L_1 = 0 \quad \text{where} \quad \omega = aL_3 = L_3\left(\frac{1}{I_1} - \frac{1}{I_3}\right). \tag{254}$$

The general solutions depend on a multiplicative constant C and an additive phase δ

$$L_1 = C\cos(\omega t + \delta) \quad \text{and} \quad L_2 = C\sin(\omega t + \delta) \tag{255}$$

The motion of the angular momentum vector with respect to a frame fixed in the top is periodic in time. The component of angular momentum along the axis of the symmetric top is fixed wile the component orthogonal to it rotates at an angular speed ω. Since $L_1 = I_1\Omega_1, L_2 = I_1\Omega_2, L_3 = I_3\Omega_3$, the same holds for the angular velocity vector, it rotates about the axis of the top at the constant rate ω. In fact

$$\Omega_3 = \frac{L_3}{I_3} = \text{ constant}, \quad \text{while} \quad \Omega_1 = \frac{C}{I_1}\cos(\omega t + \delta) \quad \text{and} \quad \Omega_2 = \frac{C}{I_1}\sin(\omega t + \delta) \tag{256}$$

In particular, Ω_3 is time-independent as is L_3. Seen from the body-fixed frame, the angular velocity vector $\vec{\Omega}$ precesses about the axis of the top at the angular rate ω, sweeping out a cone of opening angle $\arctan(C/I_1\Omega_3)$. Similarly, the angular momentum vector $\vec{L}$ precesses about the axis of the top at the angular frequency ω and sweeps out a cone of opening angle $\arctan(C/L_3)$.

• Earlier we found that $L_3 = L\cos\theta$ or $\Omega_3 = \frac{L}{I_3}\cos\theta$, where the Euler angle θ is the angle between the (constant) angular momentum vector in space and the instantaneous direction of the axis of the top. So we deduce that $\theta = \arctan(C/L_3)$ is independent of time. The axis of the top precesses at a constant angle around the fixed direction of angular momentum in the lab frame. If the angular velocity vector pointed along the axis of the top (i.e., if $C = 0$), then we would say that the top simply spun on its axis. For $C \neq 0$, the above solution describes the force-free motion of a top that is spinning on its own axis and precessing about an axis fixed in space.

• Furthermore, the square of the length of the angular velocity vector is constant in time. To see this, we compute it in the co-moving frame

$$\Omega^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \frac{C^2}{I_1^2} + \frac{C^2}{I_2^2} + \frac{L_3^2}{I_3^2} = \text{ constant}. \tag{257}$$

The *magnitude* of $\mathbf{\Omega}$ is of course the same in the co-rotating and lab frames, since rotations do not alter lengths of vectors.

• Let us interpret the angular frequency ω. From the above solution, L_3 is constant while $L_1 = C\cos(\omega t + \varphi)$ and $L_2 = C\sin(\omega t + \varphi)$. So ω is the rate at which the $\mathbf{L}$ vector precesses about the z axis of the co-moving frame. Above we found

$$\omega = L_3\left(\frac{1}{I_1} - \frac{1}{I_3}\right) = L\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_3}\right) = -\dot{\psi}. \tag{258}$$

The last equality follows from what we found in the section on Euler angles. So $-\dot{\psi}$ is the rate at which the $\mathbf{L}$ vector precesses around the z axis of the co-rotating frame, which is also evident from fig. 7.

• The limiting case of a rigid rotator may be obtained by letting $I_3 \to 0$ holding I_1 fixed. In other words, we first let $I_2 \to I_1$ and then let $I_3 \to 0$ so that the symmetric top becomes infinitesimally thin. The only way for ω to remain finite is for $\theta \to \frac{\pi}{2}$ in such a way that $\frac{\cos\theta}{I_3}$ approaches a finite limit. Thus we find that the axis of a rigid rotator must always be perpendicular to the angular momentum in space. Since the latter is a fixed vector $\mathbf{L} \propto \hat{Z} \propto \mathbf{\Omega}$, the axis of the rotator sweeps out a disc in the XY plane. What is more, the limiting value of $\cos\theta/I_3$ must be zero. This is because $\frac{\cos\theta}{I_3} = \frac{\Omega_3}{L}$, and $\Omega_3 = 0$ since x_3 is along the axis of the rotator, which is always in a plane orthogonal to $\mathbf{\Omega}$. It follows that in this limit, $\omega \to 0$ so that Ω_1, Ω_2 are both constants. We may choose $\hat{x}$ along $\hat{Z}$ and $\hat{y}$ in the plane swept out by the axis of the rotator which points along $\hat{z}$. This corresponds to the choice of phase $\delta = 0$. Then $\Omega_1 = C/I_1 = \Omega$, $C = L$ and $\Omega_2 = \Omega_3 = 0$.

8.9 Ellipsoid of inertia and qualitative description of free motion of rigid body

• The rotational kinetic energy $T = \frac{1}{2}I_{ij}\Omega_i\Omega_j$ takes a simple form in the principal axis basis

$$T = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}. \tag{259}$$

• For torque free motion, we use Euler's equations to show that the rotational kinetic energy and square of angular momentum $L^2 = L_1^2 + L_2^2 + L_3^2$ are constant in time.

$$\dot{H} = \frac{L_1\dot{L}_1}{I_1} + \frac{L_2\dot{L}_2}{I_2} + \frac{L_3\dot{L}_3}{I_3} = -L_1L_2L_3\left(\frac{a_{23}}{I_1} + \frac{a_{31}}{I_2} + \frac{a_{12}}{I_3}\right) = 0. \tag{260}$$

Similarly, we show

$$\frac{1}{2}\frac{dL^2}{dt} = L_1\dot{L}_1 + L_2\dot{L}_2 + L_3\dot{L}_3 = -L_1(a_{23}L_2L_3) - L_2(a_{31}L_3L_1) - L_3(a_{12}L_1L_2) = -L_1L_2L_3(a_{23} + a_{31} + a_{12}) = 0.$$

• Thus we have two conserved quantities. We may use them to visualise the trajectories in the phase space of angular momenta. The phase space is $\mathbb{R}^3$ whose coordinates are L_1, L_2, L_3. The conservation of total angular momentum implies that the trajectory must lie on the angular momentum sphere $L_1^2 + L_2^2 + L_3^2 = L^2$. The radius of the sphere is determined by the initial magnitude of the angular momentum vector. In addition, conservation of energy implies that the trajectory must lie on an ellipsoid of constant energy $H = \sum_i \frac{L_i^2}{2I_i} = E$. The three semi-axes

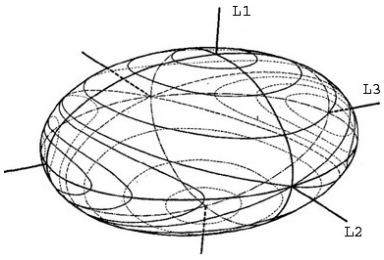


Figure 8: Trajectories of Euler equations on an energy level surface (from Landau & Lifshitz)

of this *ellipsoid of inertia* are $\sqrt{2EI_i}$ for $i = 1, 2, 3$. E is of course determined by the initial conditions. Without loss of generality let us assume that $I_1 \leq I_2 \leq I_3$ so that $I_1^{-1} \geq I_2^{-1} \geq I_3^{-1}$. Then we see that the energy satisfies the following inequalities

$$\frac{L^2}{2I_3} \leq E \leq \frac{L^2}{2I_1}. \tag{261}$$

As a consequence

$$\sqrt{2EI_1} \leq L \leq \sqrt{2EI_3}. \tag{262}$$

These inequalities imply that the energy ellipsoid and angular momentum sphere always have non-empty intersection. In other words, the radius of the angular momentum sphere lies between the smallest and largest semi-axes of the ellipsoid of inertia. The two quadratic surfaces generically intersect along a (union of) closed curves or points. Each such intersection set is a union of possible trajectories of the angular momentum vector, for given total angular momentum and energy.

• Let us get a qualitative picture of the types of curves traced out by the tip of the angular momentum vector. There are six stationary points corresponding to rotation about the principal axes of inertia: $L_{1,2,3} = \pm\sqrt{2EI_{1,2,3}}$ with others vanishing and with energy $E = \frac{L^2}{2I_{1,2,3}}$. Check that these are solutions of the Euler equations.

• More generally, suppose we keep the energy E fixed and imagine varying the magnitude of total angular momentum L. When $L^2 = 2EI_1$, there are two points of intersection, at $L_1 = \pm\sqrt{2EI_1}, L_2 = L_3 = 0$. So the angular momentum vector is static in the co-moving frame. It always points along (or opposite to) the principal axis (x-axis) corresponding to the smallest principal moment of inertia I_1. It follows that $\Omega_1 = \sqrt{\frac{2E}{I_1}}, \Omega_2 = \Omega_3 = 0$ so the angular velocity is constant and always along the x principal axis of the body-fixed frame. The locus of the angular velocity vector is called a polhode curve. Here the polhodes are a pair of points. We will see now that this motion is stable in the sense that a small change in energy/angular momentum results in a trajectory that always remains close to this one.

• As the angular momentum is increased slightly, the angular momentum sphere intersects the inertia ellipsoid in a pair of small closed curves encircling the x axis. So the static solutions of the previous paragraph (rotation of the body about the $\pm x$-axis) are stable. The present solutions correspond to a time-dependent $\mathbf{L}$ in the body fixed frame, whose terminus precesses around the $\pm x$-principal axis. Thus, the $\mathbf{L}$ vector relative to the body is periodic in time, it sweeps out a conical surface. The angular momentum vector relative to space is of course constant.

• As $L^2 \to 2EI_2$, the curves of intersection become larger and at $L^2 = 2EI_2$, are a pair of large ellipses that intersect at the points where the inertia ellipsoid meets the y axis. $L_2 = \pm\sqrt{2EI_2}, L_1 = L_3 = 0$ are static solutions corresponding to rotation about the $\pm y$-principal axis corresponding to the middle principal moment of inertia. However, rotation about the middle principal axis is unstable, since a small change in E or L^2 results in a trajectory that isn't always close to $L_2 = \pm\sqrt{2EI_2}$. The complement of these two static solutions in the pair of intersecting ellipses results in 4 separatrices. They separate the regions of phase space corresponding (broadly) to oscillations about rotation about the largest and smallest principal axes.

• As L^2 goes from $2EI_2$ to $2EI_3$, the curves of intersection shrink to a pair of curves around the z-axis and eventually end at the pair of static solutions $L_1 = L_2 = 0, L_3 = \pm\sqrt{2EI_3}$ corresponding to stable rotation about the $\pm z$-axis.

• Almost all trajectories are closed curves (except the separatrices), so the motion on phase space of angular momenta is periodic almost always. Static solutions have a period zero (on the angular momentum phase space). One can often interpret separatrices as trajectories whose period has gone to infinity.

• A rapidly spun tennis racquet (or chalkboard duster) displays the above stability while rotating about its principal axes corresponding to its largest and smallest principal moments of inertia. The above instability manifests itself when we try to spin the racquet about its middle principal axis. Of course, a spinning tennis racquet is not free but subject to the gravitational force of the earth. But if the kinetic energy of rotation is large, we would expect to be able to ignore the effects of gravity on the qualitative rotational behavior discussed above.