Problems on Quantum Theory of Scattering
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Problems concerning basic theoretical developments and examples are marked $\langle B\rangle$ and $\langle BE\rangle$. They would be useful in following the lectures.

Probability current in 1D scattering

1. $\langle B\rangle$ Consider scattering in 1D with asymptotic wave function $\psi(x) = Ae^{ikx} + B^{-ikx}$ as $x \to -\infty$. Show that the **probability current density** $j(x,t) = \frac{\hbar}{2mi}(\psi^* \partial_x \psi - \partial_x \psi^* \psi)$ as $x = -\infty$ is sum of the probability current densities of the incoming and outgoing waves: i.e., show that $j = j_A + j_B$ and give formulae for $j_{A,B}$ bearing in mind that they are vectors.

   S-Matrix for scattering in 1D

2. $\langle BE\rangle$ Consider **scattering from an attractive δ well** in one dimension, $H = \frac{p^2}{2m} - g\delta(x)$.

   (a) Find the S-matrix. You may use the known results for the transmitted and reflected amplitudes t and r for the standard scattering problems.

   (b) Verify that the S-matrix for the delta potential well is unitary.

   (c) Find the pole(s) of the S-matrix in the complex k-plane and compare the energy $E = \hbar^2 k^2/2m$ at the pole(s) with the energies of the bound states in this potential.

3. $\langle B\rangle$ **Unitarity of S-matrix** Consider the 1d scattering problem for an asymptotically vanishing real potential $V(x)$ with asymptotic amplitudes

$$\psi(x) \to \begin{cases} Ae^{ikx} + Be^{-ikx} & as\ \ x \to -\infty \\ Ce^{ikx} + De^{-ikx} & as\ \ x \to +\infty \end{cases} \tag{1}$$

   Show that the S-matrix is unitary, i.e.,

$$\left\langle \begin{pmatrix} A \\ D \end{pmatrix}, \begin{pmatrix} A' \\ D' \end{pmatrix} \right\rangle = \left\langle S \begin{pmatrix} A \\ D \end{pmatrix}, S \begin{pmatrix} A' \\ D' \end{pmatrix} \right\rangle \tag{2}$$

   Hint: Consider the Wronskian $W(\psi_1^*(x), \psi_2(x))$ where ψ_1, ψ_2 are two scattering eigenstates with the same energy E

$$\psi_1(x) \to \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{as}\ \ x \to -\infty \\ Ce^{ikx} + De^{-ikx} & \text{as}\ \ x \to +\infty \end{cases}, \quad \psi_2(x) \to \begin{cases} A'e^{ikx} + B'e^{-ikx} & \text{as}\ \ x \to -\infty \\ C'e^{ikx} + D'e^{-ikx} & \text{as}\ \ x \to +\infty. \end{cases} \tag{3}$$

   Scattering in 3D

4. $\langle B\rangle$ Consider scattering in 3d against a potential $V(r)$. Calculate the gradient of the **scattered wave** $\psi(\vec{r}) = f(\theta,\phi)\frac{e^{ikr}}{r}$ and find its leading behavior as $r \to \infty$. Also show that the scattered probability current as $r \to \infty$ is given by $j_{sc} = (\hbar k|f|^2/m)\hat{r}/r^2$.

5. $\langle B\rangle$ **Spherical Bessel equation:** radial equation for free particle energy eigenstates with angular momentum quantum number l.

   (a) Show that the radial equation for free particle energy eigenstates

$$\frac{-\hbar^2}{2m}\frac{1}{r}\frac{d^2(rR)}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2}R = \frac{\hbar^2 k^2}{2m}R \tag{4}$$

   becomes the spherical Bessel equation

$$-\frac{d^2 u_l(\rho)}{d\rho^2} + \frac{l(l+1)}{\rho^2}u_l = u_l \tag{5}$$

   where $u = rR$ and $\rho = kr$.

   (b) Find the two linearly independent solutions of this equation for $l = 0$.

6. $\langle B\rangle$ **Method of Infeld for j_l and n_l:** The radial Schrodinger eigenvalue problem for a free particle in spherical coordinates (for $u(\rho) = rR(r)$ where $\rho = kr$) takes the form of the spherical Bessel equation

$$\left(-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2}\right)u_l(\rho) = u_l(\rho). \tag{6}$$

We seek to build up the solutions for $l \geq 1$ using 'raising operators' applied to the orthogonal solutions for $l = 0$, namely $j_0(\rho) = u_0/\rho = \sin\rho/\rho$ and $n_0(\rho) = u_0/\rho = -\cos\rho/\rho$.

   (a) In what sense are $j_0(\rho)$ and $n_0(\rho)$ orthogonal, and why is this reasonable?

   (b) Suppose we define the 'lowering operator' $d_l = \frac{d}{d\rho} + \frac{l+1}{\rho}$. Find the raising operator $d_l^\dagger$ which is its Hermitian adjoint with respect to the inner product on the solutions $u_l(\rho)$.

   (c) Show that the spherical Bessel equation can be 'factorized' as $d_l d_l^\dagger u_l = u_l$.

   (d) Show also that $d_l^\dagger d_l = d_{l+1} d_{l+1}^\dagger$.

   (e) Use this to deduce that $\left(d_{l+1}d_{l+1}^\dagger\right)(d_l^\dagger u_l) = d_l^\dagger u_l$. What is the use of this result?

   (f) Suppose we normalize so that $d_l^\dagger u_l = u_{l+1}$, then show that $R_{l+1} = \left(-\frac{d}{d\rho} + \frac{l}{\rho}\right)R_l(\rho)$.

   (g) Further simplify this result to conclude that $R_l = (-\rho)^l\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l R_0(\rho)$. Hint: First show that $\frac{R_{l+1}}{\rho^{l+1}} = \left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\frac{R_l}{\rho^l}$.

   (h) Use this to find the spherical Bessel & Neumann functions $j_1(\rho), j_2(\rho), n_1(\rho), n_2(\rho)$.

   (i) Try to argue why they must have the asymptotic behaviors

$$j_l(\rho) \to \frac{\sin\left(\rho - \frac{l\pi}{2}\right)}{\rho} \quad \text{and} \quad n_l(\rho) \to -\frac{\cos\left(\rho - \frac{l\pi}{2}\right)}{\rho} \quad \text{as } \rho \to \infty. \tag{7}$$

(j) Show that the asymptotic behavior (as $\rho \to \infty$) of the spherical Hankel functions $h_l^{\pm}(\rho) = j_l(\rho) \pm i\, n_l(\rho)$ is given by $(\mp i)^{l+1} e^{\pm i\rho}/\rho$.

Partial wave expansion

7. $\langle B \rangle$ Use the partial wave expansion in terms of phase shifts to relate the total cross section σ for scattering by a spherically symmetric potential $V(r)$ to the imaginary part of the forward scattering amplitude $f(\theta = 0)$. Hint: To find $f(0)$ use $P_l(x) = \frac{1}{2^l l!} \frac{d^l (x^2-1)^l}{dx^l}$ where $x = \cos\theta$, work out the cases $l = 0, 1, 2$ and observe a pattern.

8. $\langle B \rangle$ We wish to find the coefficients C_l appearing in the expansion of a plane wave as a linear combination of spherical waves:

$$e^{ikr\cos\theta} = \sum_{l'=0}^{\infty} C_{l'} (2l'+1) P_{l'}(\cos\theta) j_{l'}(kr). \tag{8}$$

(a) If $\rho = kr$, **show that** C_l must satisfy the following identity for each $l = 0, 1, 2, \ldots$ and $\rho \geq 0$:

$$\int_{-1}^{1} e^{i\rho x} P_l(x)\, dx \;=\; 2\, C_l\, j_l(\rho) \quad \text{where } \; x = \cos\theta. \tag{9}$$

(Hint: Use the orthogonality relation $\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'}$)
The above identity must in particular be true as $\rho \to 0$. So we will compare the leading behavior of both sides as $\rho \to 0$ to extract the constants C_l. For this we use the formulae

$$P_l(x) = \frac{1}{(2l)!!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad \text{and} \quad j_l(\rho) \to \frac{\rho^l}{(2l+1)!!} \;\text{ as }\; \rho \to 0. \tag{10}$$

(b) Integrating by parts, express the LHS of (9) in terms of the l^{th} derivative of $e^{i\rho x}$.

(c) Find $n_{\min}$ and show that

$$\frac{d^l}{dx^l} e^{i\rho x} = \sum_{n=n_{\min}}^{\infty} \frac{i^n \rho^n}{(n-l)!} x^{n-l}. \tag{11}$$

(d) Find the leading behavior of $\frac{d^l}{dx^l} e^{i\rho x}$ for small ρ.

(e) Use this to show the behavior of the LHS of (9) for small ρ is given by

$$\int_{-1}^{1} e^{i\rho x} P_l(x)\, dx \;\to\; \frac{(-1)^l i^l \rho^l}{(2l)!!} \int_{-1}^{1} (x^2 - 1)^l\, dx. \tag{12}$$

(f) Comparing with the behavior of the RHS, find C_l given that

$$\int_0^{\frac{\pi}{2}} \sin^{2l+1}\theta\, d\theta = \frac{(2l)!!}{(2l+1)!!}. \tag{13}$$

9. $\langle BE \rangle$ Consider scattering of plane waves of wave number k (incident along z from $z = -\infty$) against an **infinitely hard sphere** potential:

$$V(r) = \begin{cases} \infty & \text{for } \; r \leq a \\ 0 & \text{for } \; r > a \end{cases} \tag{14}$$

(a) Show that the partial wave phase shifts are given by

$$\delta_l = \arctan \frac{j_l(ka)}{n_l(ka)}. \tag{15}$$

(b) Find the S-wave phase shift δ_0 and partial wave amplitude a_0.

(c) Show that the S-wave scattering cross section is given by

$$\sigma_0(k) = \frac{4\pi}{k^2} \sin^2(ka). \tag{16}$$

(d) Find the S-wave cross section in the limit of very low energies $(k \to 0)$ and compare it with the classical cross section.

(e) Show that the S-wave scattering length $\alpha = a$.

Rutherford cross section for Coulomb scattering

10. $\langle B \rangle$ The differential cross section for Coulomb scattering of a projectile of charge q and incident angular wave number k by a fixed charge Q located at $r = 0$ is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mQq}{16\pi\epsilon_0 \hbar^2 k^2} \right)^2 \frac{1}{\sin^4(\theta/2)}. \tag{17}$$

(a) Check that the differential cross section has the correct dimensions.

(b) Find the total cross section for Coulomb scattering. Does it matter whether the charges are of the same or opposite signs?

(c) In a polar plot, roughly sketch the angular distribution of scattered particles (differential cross section). In which direction do most of the scattered particles go?

(d) What is the classical limit of the differential cross section for Coulomb scattering?

Born approximation

11. $\langle B \rangle$ Show that $\nabla^2(\frac{1}{r}) = -4\pi\delta^3(\vec{r})$. The Laplacian in spherical polar coordinates is

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{18}$$

Hint: First consider $r > 0$. To deal with $\mathbf{r} = 0$, integrate both sides and use Gauss' divergence theorem.

12. $\langle B \rangle$ Prove the identity $|\mathbf{r} - \mathbf{r}'| = r - \hat{r} \cdot \mathbf{r}' + O(\frac{r'^2}{r^2})$ for $|\frac{r'}{r}| \ll 1$. Hint: Compute $(\mathbf{r} - \mathbf{r}')^2$.

13. $\langle B \rangle$ Derive the formula $q = 2k \sin\frac{\theta}{2}$, for the magnitude q of the 'momentum transfer' $\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i$. Here $\mathbf{k_i}$ and k are the wave vector and wave number $k = |\mathbf{k}_i|$ of the incoming plane wave and $\mathbf{k}_f = k\hat{r}$ is the 'outgoing wave vector' in the direction of interest. θ is the angle between incident and 'scattered' directions $\mathbf{k}_i$ and $\mathbf{k}_f$. This formula is used in obtaining the Born approximation for spherically symmetric potentials. Hint: Draw a figure.

14. $\langle B \rangle$ Show that

$$\int_0^\infty dr\, e^{-\mu r} \sin qr = \frac{q}{q^2 + \mu^2} \quad \text{for} \quad \mu > 0. \tag{19}$$

This integral with q the magnitude of the momentum transferred appears in the Born approximation for the scattering amplitude in a screened Coulomb potential $\propto -e^{-\mu r}/r$.

15. $\langle BE \rangle$ Consider **scattering from a finite spherical barrier** $V(r) = V_o \theta(r < a)$ with $V_o > 0$, at relatively high energies $E > V_o$ where the Born approximation may be applicable. θ is the Heaviside step function.

    (a) Find the scattering amplitude $f(\theta, \phi)$ and total cross-section at *zero momentum transfer* in the Born approximation.

    (b) Generally speaking, under what circumstances (concerning physical/geometric parameters) might the momentum transfer $\vec{q}$ be small?

    (c) Evaluate the scattering length α_{Born} at zero $\vec{q}$ in the Born approximation, check it has the correct dimensions and expected sign.

    i. The S-wave phase shift for small δ_0 and ka is $\delta_0 \approx ka \left(\frac{\tanh \kappa a}{\kappa a} - 1 \right)$ for $\kappa^2 = \frac{2m}{\hbar^2}(V_o - E)$. Find the corresponding scattering length α. Compare this α with α_{Born} in the limit of small V_0, where the Born approximation may be expected to be more trustworthy.

    (d) Evaluate the scattering amplitude $f(\theta)$ and differential cross section $\frac{d\sigma(\theta)}{d\Omega}$ in the Born approximation (for *arbitrary momentum transfer*). Notice the partial resemblance to the Rutherford differential cross section.

    (e) Check that in the limit of zero momentum transfer, the scattering amplitude matches the result of 15a.

    (f) What is the Rutherford differential cross section in the limit of zero momentum transfer?

    (g) On qualitative physical grounds, argue whether the total cross section for the potential $V(r) = V_o \theta(r < a)$ is finite or infinite.

    (h) Use the above Born approximation for $\frac{d\sigma(\theta)}{d\Omega}$ to argue whether the total cross section for $V(r)$ is finite or infinite. Argue based on the angular dependence of the differential cross section in the appropriate directions, there is no need to evaluate the integral exactly.