Quantum Mechanics 3, Spring 2012 CMI
Problem set 7
Due by beginning of class on Monday Mar 5, 2012
BCH formula for x and p, SHO

1. $\langle 6 \rangle$ Consider the function $f(t) = e^{tA} B e^{-tA}$ where A, B are a pair of operators (e.g. position and momentum or creation and annihilation operators etc.). t is a parameter which could be a time interval or a spatial interval for instance.

   (a) $\langle 2 \rangle$ Show that $f'(0) = [A, B],\ f''(0) = [A, [A, B]]$ etc.

   (b) $\langle 1 \rangle$ Deduce that
   $$f(t) = B + [A, B]t + [A, [A, B]]\frac{t^2}{2} + \cdots \qquad (1)$$

   (c) $\langle 1 \rangle$ What does this formula reduce to if A and B commute with their commutator?

   (d) $\langle 2 \rangle$ Apply this to the case $A = ip$ and $B = x$ and $t = a/\hbar$. Give a physical interpretation of the resulting formula.

2. $\langle 10 \rangle$ Define the function of time $U(t) = e^{tA} e^{tB}$.

   (a) $\langle 2 \rangle$ Use (1) to show that $U(t)$ is the time evolution operator for a certain hamiltonian:
   $$\dot{U}(t) = H(t)U(t) \quad \text{where} \quad H(t) = A + B + t[A, B] + \frac{t^2}{2!}[A, [A, B]] + \cdots \qquad (2)$$

   (b) $\langle 2 \rangle$ Write a formula for $U(t)$ in terms of time-ordered exponentials.

   (c) $\langle 2 \rangle$ Calculate $[H(t), H(t')]$ assuming A and B commute with their commutator.

   (d) $\langle 2 \rangle$ Use the previous result to simplify the time-ordered exponential formula for $U(t)$ assuming A and B commute with their commutator.

   (e) $\langle 2 \rangle$ Assuming A and B commute with their commutator, show that
   $$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}. \qquad (3)$$

3. $\langle 8 \rangle$ Consider the simple harmonic oscillator, $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$. Evaluate the classical action $S[x]$ for the unique trajectory connecting x_i, t_i to x_f, t_f assuming $t_f - t_i = T \neq \frac{n\pi}{\omega}$ for any integer n. For uniformity of notation, recall that the solution to Newton's equation is $x(t) = a\cos\omega t + b\sin\omega t$

   $$\text{with} \quad a = \frac{s_f x_i - s_i x_f}{s_{f-i}}, \quad b = \frac{c_i x_f - c_f x_i}{s_{f-i}}, \quad \text{where } s_i = \sin\omega t_i,\ \ s_{f-i} = \sin\omega(t_f - t_i) \text{ etc.} \qquad (4)$$

   Due to time-translation invariance, you may take $t_i = 0$ without loss of generality. Does this system have space translation invariance? May we take $x_i = 0$ without loss of generality? Hints: Use the abbreviation $c = \cos\omega T, s = \sin\omega T$ where convenient. The answer is
   $$S[x] = \frac{m\omega}{2s}[(x_i^2 + x_f^2)c - 2x_i x_f]. \qquad (5)$$