Quantum Mechanics 3, Spring 2012 CMI
Problem set 6
Due by beginning of class on Monday Feb 13, 2012
Time evolution operator, Hamilton's principle

1. $\langle 3 \rangle$ Suppose the time-dependent hamiltonian of a system $H(t)$ is *hermitian* at all times and is separable as a product of a complex function of time $c(t)$ and a time-independent operator K (not-necessarily hermitian)
$$H(t) = c(t)K \tag{1}$$
Show that we can always define a new real function of time $h(t)$ and a new hermitian operator H such that $H(t) = h(t)H$. Express $h(t)$ and H in terms of $c(t)$ and K and any other appropriate quantities.

2. $\langle 5 \rangle$ Consider the functional equation for a complex-valued function of one real variable $f(t+s) = f(t)f(s)$ subject to the initial condition $f(0) = 1$. Find all possible solutions of this functional equation and give a quasi-physical interpretation of any new quantities that appear in your answer.

3. Consider the classical mechanics of a particle of mass m in an SHO potential $V(x) = \frac{1}{2}m\omega^2x^2$. Suppose $x(t)$ is a classical trajectory between $x_i(t_i)$ and $x_f(t_f)$ and let $x(t) + \delta x(t)$ be a neighboring path with $\delta x(t_i) = \delta x(t_f) = 0$.

   (a) $\langle 2 \rangle$ Write the classical action of the path $x + \delta x$ as a Taylor polynomial in δx
   $$S[x + \delta x] = S_0 + S_1 + \cdots + S_k \quad \text{where } S_n \text{ are } n^{\text{th}} \text{ order in } \delta x. \tag{2}$$
   (b) $\langle 3 \rangle$ Show that Hamilton's principle of least action implies Newton's equation for this particle.
   (c) $\langle 4 \rangle$ Find the trajectory joining $x(t_i) = x_i$ and $x(t_f) = x_f$ using abbreviations like $s_{i,f} = \sin\omega t_{i,f}$.
   (d) $\langle 2 \rangle$ Express S_2 in the form given below and extract the operator A:
   $$S_2 = \int_{t_i}^{t_f} \delta x(t)\, A\, \delta x(t)\, dt \equiv \langle \delta x | A | \delta x \rangle. \tag{3}$$
   (e) $\langle 2 \rangle$ Show that A is bounded below, in the sense that
   $$\langle \delta x | A | \delta x \rangle \geq -\frac{1}{2}m\omega^2 ||\delta x||^2 \quad \text{where} \quad ||\delta x||^2 = \int_{t_i}^{t_f} \delta x(t)\, \delta x(t)\, dt. \tag{4}$$
   (f) $\langle 1 \rangle$ For what values of κ is $x(t) + \delta x(t)$ a legitimate neighboring path for the variation
   $$\delta x(t) = \epsilon\, \sin\kappa(t - t_i)\ ? \tag{5}$$
   (g) $\langle 2 \rangle$ Evaluate $S_2[\delta x]$ for all the allowed values of κ.
   (h) $\langle 1 \rangle$ Take $T = t_f - t_i = 10$s and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *less* than that of $x(t)$.
   (i) $\langle 1 \rangle$ Take $T = t_f - t_i = 10$s and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *more* than that of $x(t)$.
   (j) $\langle 2 \rangle$ Explain the significance of the above two examples with regard to Hamilton's least action principle.