Quantum Mechanics 3, Spring 2012 CMI
Problem set 4
Due by beginning of class on Monday Jan 30, 2012
Time evolution due to time-dependent hamiltonian

1. ⟨3⟩ Calculate the commutator of the hamiltonians $H_{1,2}$ of a pair of 1d simple harmonic oscillators with distinct frequencies $\omega_1 \neq \omega_2$. Does it matter (to the commutator) whether the masses are equal?

2. ⟨3⟩ Show that $[H_1, H_2] \neq 0$ by exhibiting a state on which this commutator is non-zero.

3. ⟨3⟩ Show that inner products are preserved by Schrödinger time evolution by a time-dependent hamiltonian $H(t)$. Describe the set up, before calculating anything.

4. ⟨14⟩ Consider a toy quantum system whose Hilbert space is one dimensional and whose hamiltonian $H(t)$ is time-dependent.

   (a) ⟨3⟩ What is $[H(t), H(t')]$?

   (b) ⟨1⟩ Solve the Schrödinger initial value problem $i\hbar\dot{\psi}(t) = H(t)\psi(t)$ for $\psi(t)$ with initial condition $\psi(0)$ and extract the time evolution operator $U(t)$.

   (c) ⟨1⟩ Expand $U(t)$ in an exponential series.

   (d) ⟨3⟩ What do you think the radius of convergence of the above series as a function of t is? Why? You may assume that the hamiltonian is bounded $|H(t')| \leq E$ for $0 \leq t' \leq t$.

   (e) ⟨3⟩ Show that the second term in the series can be expressed as

$$\frac{1}{(i\hbar)^2 2!}\left(\int_0^t H(t')dt'\right)^2 = \frac{1}{(i\hbar)^2}\int_0^t dt'\, H(t')\int_0^{t'} dt''\, H(t'') \tag{1}$$

   (f) ⟨3⟩ Argue that the series for $U(t)$ can be expressed as

$$U(t) = \sum_{n=0}^{\infty}\frac{1}{(i\hbar)^n}\int_{t\geq t_1\geq\ldots\geq t_n\geq 0} dt_1\cdots dt_n\, H(t_1)H(t_2)\ldots H(t_n). \tag{2}$$

5. ⟨7⟩ Let us illustrate the idea that even though the hamiltonian may be periodic with period T, the wave function need not be. Consider the toy Schrödinger initial value problem on a 1d Hilbert space

$$i\dot{\psi} = h(t)\,\psi(t), \quad \text{with periodic hamiltonian} \quad h(t+T) = h(t) \tag{3}$$

   subject to the initial condition $\psi(0)$.

   (a) ⟨1⟩ Find the solution $\psi(t)$ to the initial value problem.

   (b) ⟨2⟩ Find the condition on $h(t)$ for $\psi(t)$ to be periodic with the same period T as the hamiltonian.

   (c) ⟨1⟩ Give an example of an $h(t)$ that has the above-determined property.

   (d) ⟨3⟩ Discuss the example $h(t) = \sin^2 t$ and say how it illustrates the idea mentioned in the problem.