Quantum Mechanics 3, Spring 2012 CMI
Problem set 3
Due by beginning of class on Monday Jan 23, 2012
Spin in a rotating magnetic field & Normal modes

Consider a spin magnetic moment at rest at the origin subject to magnetic field of fixed magnitude whose direction sweeps out a cone of opening angle θ at a constant angular speed ω

$$\vec{B}(t) = B\hat{n}(t) = B\left[\hat{z}\cos\theta + \hat{x}\sin\theta\cos\omega t + \hat{y}\sin\theta\sin\omega t\right]. \tag{1}$$

The hamiltonian is $H(t) = \omega_l \vec{S}\cdot\hat{n}(t)$ where $\omega_l = eB/m$ is the Larmor frequency. We wrote the spin wave function at time t in terms of the dynamical phases $\theta^D_\pm = \mp\omega_l t/2$

$$\psi(t) = c_+\psi_+ e^{i\theta^D_+} + c_-\psi_- e^{i\theta^D_-} \tag{2}$$

and the instantaneous eigenstates of $H(t)$

$$\psi_+(t) = \begin{pmatrix} \cos(\theta/2) \\ e^{i\omega t}\sin(\theta/2) \end{pmatrix} \quad \text{and} \quad \psi_-(t) = \begin{pmatrix} e^{-i\omega t}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}. \tag{3}$$

We showed that $c_\pm(t)$ satisfy the coupled system of ODEs with time-dependent coefficients

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = -\frac{\hbar\omega}{2}\begin{pmatrix} -2\sin^2(\theta/2) & e^{i(\omega_l-\omega)t}\sin\theta \\ e^{-i(\omega_l-\omega)t}\sin\theta & 2\sin^2(\theta/2) \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}. \tag{4}$$

We endeavor to solve this system exactly in two steps: make the coefficients in the ODEs time independent and then transform to *normal modes*.

1. $\langle 4\rangle$ Define the new variables $d_\pm = c_\pm e^{\mp i\phi/2}$ where $\phi = (\omega_l - \omega)t$. Show that in terms of $d_\pm(t)$ we get a system of coupled 1st order ODEs with *time-independent* coefficients

$$-2i\frac{\partial}{\partial t}\begin{pmatrix} d_+ \\ d_- \end{pmatrix} = A\begin{pmatrix} d_+ \\ d_- \end{pmatrix}, \quad \text{with} \quad A = \begin{pmatrix} -\omega_l + \omega\cos\theta & \omega\sin\theta \\ \omega\sin\theta & \omega_l - \omega\cos\theta \end{pmatrix}. \tag{5}$$

2. $\langle 2\rangle$ What is the physical significance of the hermiticity and tracelessness of A?

3. $\langle 2\rangle$ Find the eigenvalues $\lambda_\pm$ of A with $\lambda_+ > 0$. Express $\lambda_\pm$ in terms of $\lambda = \sqrt{\omega_l^2 - 2\omega_l\omega\cos\theta + \omega^2}$.

4. $\langle 3\rangle$ Find the corresponding eigenvectors $Av_\pm = \lambda_\pm v_\pm$ and show that they can be expressed as

$$S = \begin{pmatrix} v_+ & v_- \end{pmatrix} = \begin{pmatrix} \omega\sin\theta & \omega\sin\theta \\ \omega_l - \omega\cos\theta + \lambda & \omega_l - \omega\cos\theta - \lambda \end{pmatrix}. \tag{6}$$

5. $\langle 3\rangle$ Explain why the similarity transformation S is not orthogonal and check that the inverse matrix is

$$S^{-1} = \frac{1}{2\lambda\omega\sin\theta}\begin{pmatrix} \lambda - \omega_l + \omega\cos\theta & \omega\sin\theta \\ \lambda + \omega_l - \omega\cos\theta & -\omega\sin\theta \end{pmatrix} \tag{7}$$

6. $\langle 1\rangle$ Write the solutions to the eigenvalue problem for A in terms of S, S^{-1} and $\Lambda = \text{diag}(\lambda_+, \lambda_-)$.

7. $\langle 3 \rangle$ Briefly explain the strategy 'passage to normal modes' by which the system (5) is to be solved.

8. $\langle 2 \rangle$ Find the system of ODEs satisfied by $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$ where $d = Sf$. Why are $f_\pm$ good variables?

9. $\langle 2 \rangle$ Solve the Schrödinger initial value problem for $f_\pm$, express the answer in terms of $f_\pm(0)$ and λ.

10. $\langle 2 \rangle$ Work backwards to construct the general solution to the IVP for $c_\pm$, show that you get

$$\begin{aligned} c_+(t) &= e^{i\phi(t)/2}\omega \sin\theta \left[f_+(0)e^{i\lambda t/2} + f_-(0)e^{-i\lambda t/2} \right] \quad \text{and} \\ c_-(t) &= e^{-i\phi(t)/2} \left[(\omega_l - \omega\cos\theta + \lambda)\, f_+(0)\, e^{i\lambda t/2} + (\omega_l - \omega\cos\theta - \lambda)\, f_-(0)\, e^{-i\lambda t/2} \right] \end{aligned} \tag{8}$$

11. $\langle 4 \rangle$ Now suppose the initial condition is spin up $\psi(0) = \psi_+(0)$. From here on we work with this initial condition. Find the corresponding initial values $f_\pm(0)$ in terms of $\lambda, \omega, \omega_l$ and θ.

12. $\langle 3 \rangle$ For this initial condition, show that

$$c_+(t) = e^{i\phi/2} \left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i}{\lambda}(\omega\cos\theta - \omega_l)\sin\left(\frac{\lambda t}{2}\right) \right] \quad \text{and} \quad c_-(t) = \frac{i\omega}{\lambda} e^{-i\phi/2} \sin\left(\frac{\lambda t}{2}\right) \sin\theta \tag{9}$$

13. $\langle 2 \rangle$ For a spin up initial condition deduce that

$$\psi(t) = \left[\cos\frac{\lambda t}{2} + \frac{i}{\lambda}(\omega\cos\theta - \omega_l)\sin\frac{\lambda t}{2} \right] e^{-\frac{i\omega t}{2}} \psi_+(t) + \left[\frac{i\omega}{\lambda}\sin\theta\sin\frac{\lambda t}{2} \right] e^{\frac{i\omega t}{2}} \psi_-(t). \tag{10}$$

14. $\langle 1 \rangle$ Find the first two leading terms in an expansion of λ around the adiabatic limit $\omega/\omega_l \to 0$.

15. $\langle 4 \rangle$ Now take the adiabatic limit $\omega/\omega_l \to 0$ in $\psi(t)$ above and show that

$$\psi(t) = e^{i\theta_+^D} e^{i\gamma_+} \psi_+(t) \tag{11}$$

for appropriate dynamical and geometric phases to be identified.