Quantum Mechanics 3, Spring 2012 CMI

Problem set 13

Due by beginning of class on Monday Apr 16, 2012

Dirac equation

1. Consider an infinitesimal Lorentz transformation $x' = \Lambda x$, where $\Lambda^\mu_{\ \nu} \approx \delta^\mu_\nu + \omega^\mu_{\ \nu}$.

   (a) For an infinitesimal boost by velocity v ($\beta = v/c \ll 1$) along the direction x, find $\omega^\mu_{\ \nu}$ and write it as a matrix.

   (b) For the above boost, find $\omega_{\mu\nu}$, write it as a matrix and verify that it is antisymmetric. Explicity identify the value of ω_{01}.

   (c) Consider now an infinitesimal rotation $\Lambda = R_\theta$ of x, y axes clockwise by angle θ about the z-axis. Find ω^μ_ν and express it as a 4×4 matrix and identify $\omega^1_{\ 2}$.

   (d) For the above rotation, find $\omega_{\mu\nu}$, express it as a matrix and verify it is anti-symmetric, and give the value of ω_{12}.

   (e) Find the matrix $S(R_\theta) = I - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}$ that implements the above infinitesimal rotation on Dirac spinor space and express it in terms of the third Pauli matrix σ_3.

   (f) The corresponding finite rotation by angle θ is implemented by $S(R_\theta) = \exp\left[-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right]$. Find $S(R_\theta)$ and express it in terms of σ_3.

   (g) Based on the above results, find and comment on how a Dirac spinor transforms under a spatial rotation of 2π and 4π about the z-axis.

2. Suppose $V(\vec{r})$ is a real potential, $\vec{p} = -i\hbar\nabla$ the momentum operator and $\vec{\sigma}$ the Pauli matrices.

   (a) Find the adjoint of the operator $(\vec{\sigma} \cdot (\vec{p}V))(\vec{\sigma} \cdot \vec{p})$ acting on two component wave functions $\psi(\vec{r})$. Express the answer in terms of $\vec{\sigma}, \vec{p}, V$. Here $(\vec{p}V)$ denotes $-i\hbar(\nabla V)$.

   (b) Express the operator $(\vec{p}V) \cdot \vec{p} - \vec{p} \cdot (\vec{p}V)$ in as simple a form as possible.

   (c) Show that
$$\vec{\sigma} \cdot (\vec{p}V) \times \vec{p} - \vec{\sigma} \cdot \vec{p} \times (\vec{p}V) = 2\vec{\sigma} \cdot (\vec{p}V) \times \vec{p}. \tag{1}$$

3. What is the Compton wavelength of a massive particle? Give a formula and physical interpretation. Numerically, what is the Compton wavelength of an electron, as a fraction of the Bohr radius?

4. Consider an electron in a Hydrogen atom, treated in the non-relativistic context of the Schrödinger equation. Its orbital, spin and total angular momentum operators are denoted $\vec{L}, \vec{S}, \vec{J} = \vec{L} + \vec{S}$. Suppose the eigenvalues of J^2, L^2, S^2 are denoted $\hbar^2 j(j+1), \hbar^2 l(l+1), \hbar^2 s(s+1)$. Show that the factor F (defined below) may be expressed as $F = \left(j + \frac{1}{2}\right)^{-1}$, for $l \neq 0$:
$$F = \frac{1}{l + \frac{1}{2}} - \frac{j(j+1) - l(l+1) - s(s+1)}{2l\left(l + \frac{1}{2}\right)(l+1)}. \tag{2}$$