Quantum Mechanics 3, Spring 2012 CMI
Problem set 11
Due by beginning of class on Monday Apr 2, 2012
Dirac equation

1. $\langle 5 \rangle$ Recall the eigenvalue problem $(H - EI)u = 0$ for the constant Dirac spinor u that arose in our search for plane wave solutions $ue^{i(\vec{p}\cdot\vec{r}-Et)/\hbar}$ of the Dirac equation. Here $H - EI$ is the 4×4 matrix consisting of 2×2 blocks

$$H - EI = \begin{pmatrix} (mc^2 - E)I & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2+E)I \end{pmatrix} \tag{1}$$

where $\vec{p} = (p_x, p_y, p_z)$ is a constant momentum vector labeling the plane waves and σ_i are the Pauli matrices. Directly calculate the determinant of $H - E$ and thereby determine the possible energies of a plane wave with wave vector $\vec{p}/\hbar$.

ANS: Calculate the 4×4 determinant expanding along the first row. Or use the formula $\det(AB|CD) = \det(AD - BC)$ if the blocks commute. $\det(H - EI) = (E^2 - p^2c^2 - m^2c^4)^2$. So $E = E_+, E_-$ each with multiplicity two. Here $E_+ = -E_- = \sqrt{p^2c^2 + m^2c^4}$. So the spectrum is $(-\infty, -mc^2] \cup [mc^2, \infty)$ but each energy is degenerate with respect to the direction of momentum and doubly degenerate over and above that.

2. The constant spinors appearing in plane wave solutions $\Psi = u\,\psi(\vec{r})T(t)$ of the free particle Dirac equation are

$$u^{(1)} = \begin{pmatrix} \uparrow \\ \frac{c(\sigma\cdot p)}{E_+ + mc^2}\uparrow \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} \downarrow \\ \frac{c(\sigma\cdot p)}{E_+ + mc^2}\downarrow \end{pmatrix}, \quad u^{(3)} = \begin{pmatrix} \frac{c(\sigma\cdot p)}{E_- - mc^2}\uparrow \\ \uparrow \end{pmatrix}, \quad u^{(4)} = \begin{pmatrix} \frac{c(\sigma\cdot p)}{E_- - mc^2}\downarrow \\ \downarrow \end{pmatrix}. \tag{2}$$

(a) $\langle 4 \rangle$ Verify that $u^{(1)}$ and $u^{(4)}$ are eigenspinors of the Dirac hamiltonian $H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$ with appropriate eigenvalues.

ANS: They are eigenspinors with eigenvalues E_+ and E_-. Can work with 2-component blocks, no need to write out all 4 components.

(b) $\langle 4 \rangle$ Find the inner products of the eigenspinors $\langle u^{(i)}|u^{(j)}\rangle$ for $i \neq j$.

ANS: They are orthogonal. Use hermiticity of $\sigma \cdot p$ and $E_- = -E_+$.

3. Recall the local conservation law for the Dirac equation $\frac{\partial P}{\partial t} + \nabla\cdot\vec{j} = 0$, where $P(x,t) = \psi^\dagger\psi$ and $\vec{j}(x,t) = \psi^\dagger c\vec{\alpha}\psi$.

(a) $\langle 2 \rangle$ Give an interpretation of $c\vec{\alpha}$ based on an analogy with hydrodynamics.

ANS: Interpretation of $c\alpha$ as velocity of the particle.

(b) $\langle 5 \rangle$ Calculate $\frac{\partial}{\partial t}\langle\psi|x|\psi\rangle$, the time derivative of the expectation value of position x in a state $|\psi(t)\rangle$ that evolves by the Dirac equation. Comment whether the result supports the hydrodynamic interpretation.

ANS: $i\hbar\partial_t\langle x\rangle_\psi = \langle\psi|[x,H]|\psi\rangle$. Now use $[x_i, p_j] = i\hbar\delta_{ij}$ $[x, H] = [x, c\alpha\cdot p] = c\alpha[x,p] = i\hbar c\alpha$. So $\partial_t\langle x\rangle_\psi = \langle\psi|c\alpha|\psi$. This supports the hydrodynamic interpretation of $c\alpha$ as a 'velocity operator'.

4. $\langle 2 \rangle$ Find the magnitude of the gap in the spectrum of the Dirac hamiltonian for an electron, in millions of electron volts.

   ANS: $.511 * 2 = 1.022$ MeV.