Quantum Mechanics 2, Autumn 2011 CMI

Problem set 5

Due by beginning of class on Monday September 19, 2011

Addition of angular momenta, rotation of spinors, Identical particles

1. Find the 2×2 matrix representing a counter-clockwise rotation (by angle ϕ about the $\hat{n}$ direction), of the spin wavefunction of a spin-$\frac{1}{2}$ particle. Express the answer as a linear combination of the identity and Pauli matrices.

2. Show that the exchange operator acting on the Hilbert space of two identical particles is hermitian.

3. Obtain a matrix representation of the Clebsch-Gordan coefficients for the transformation from uncoupled to coupled basis of a system of two spin-half particles. Show that the matrix is unitary. Specify the uncoupled and coupled basis vectors and their orderings.

4. Suppose $\vec{J} = \vec{L} + \vec{S}$ is the sum of orbital and spin angular momentum of an electron (say in a hydrogenic atom). Does $\vec{L} \cdot \vec{S}$ commute with each of J^2, J_z, L^2 and S^2?

5. Find the matrix elements of the spin-orbit interaction between coupled basis states

$$\langle j\, m_j\, l\, s \mid f(r)\vec{L} \cdot \vec{S} \mid j'\, m_j'\, l'\, s' \rangle \tag{1}$$

   where $f(r)$ is a function of the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$ alone.

6. Consider three identical particles each of which has a translational degree of freedom (x) and a *color* degree of freedom (α). There are three values that color can take $\alpha = 1, 2, 3$ while there are infinitely many values that location x can take. The system of three particles is in a factorized state constructed from a single one-particle orbital $\phi(x)$

$$\psi(x_1, \alpha_1; x_2, \alpha_2; x_3, \alpha_3) = A \epsilon_{\alpha_1 \alpha_2 \alpha_3} \phi(x_1)\phi(x_2)\phi(x_3) \tag{2}$$

   Is this a system of bosons or fermions? Justify your answer.

7. Find the constant A so that the total wave function is normalized to one, assuming that the one particle wave function has norm $||\phi|| = 1$.