**Quantum Mechanics 2, Autumn 2011 CMI**
Problem set 13
Due by beginning of class on Monday November 21, 2011
Scattering & Perturbation Theory

1. Consider scattering from a finite spherical barrier $V(r) = V_o\theta(r < a)$ with $V_o > 0$, now at relatively high energies $E > V_o$ where the Born approximation may be applicable.

   (a) $\langle 3 \rangle$ Find the scattering amplitude $f(\theta,\phi)$ and total cross-section at *zero momentum transfer* in the Born approximation.

   (b) $\langle 2 \rangle$ Generally speaking, under what circumstances (concerning physical/geometric parameters) might the momentum transfer $\vec{q}$ be small?

   (c) $\langle 2 \rangle$ Evaluate the scattering length $\alpha_{Born}$ at zero $\vec{q}$ in the Born approximation, check it has the correct dimensions and expected sign.

      i. $\langle 4 \rangle$ The S-wave phase shift for small $\delta_0$ and $ka$ is $\delta_0 \approx ka\left(\frac{\tanh \kappa a}{\kappa a} - 1\right)$ for $\kappa^2 = \frac{2m}{\hbar^2}(V_o - E)$. Find the corresponding scattering length $\alpha$. Compare this $\alpha$ with $\alpha_{Born}$ in the limit of small $V_0$, where the Born approximation may be expected to be more trustworthy.

   (d) $\langle 4 \rangle$ Evaluate the scattering amplitude $f(\theta)$ and differential cross section $\frac{d\sigma(\theta)}{d\Omega}$ in the Born approximation (for *arbitrary momentum transfer*). Notice the partial resemblance to the Rutherford differential cross section.

   (e) $\langle 2 \rangle$ Check that in the limit of zero momentum transfer, the scattering amplitude matches the result of 1a.

   (f) $\langle 1 \rangle$ What is the Rutherford differential cross section in the limit of zero momentum transfer?

   (g) $\langle 1 \rangle$ On qualitative physical grounds, argue whether the total cross section for the potential $V(r) = V_o\theta(r < a)$ is finite or infinite.

   (h) $\langle 3 \rangle$ Use the above Born approximation for $\frac{d\sigma(\theta)}{d\Omega}$ to argue whether the total cross section for $V(r)$ is finite or infinite. Argue based on the angular dependence of the differential cross section in the appropriate directions, there is no need to evaluate the integral exactly.

2. $\langle 5 \rangle$ Consider the inhomogeneous linear equation $x = 1 + gx$ for the unknown complex variable $x$, where $g$ is some complex parameter.

   (a) $\langle 1 \rangle$ Find the exact solution $x(g)$.

   (b) $\langle 2 \rangle$ Define the sequence of approximants $x^{(n)} = 1 + gx^{(n-1)}$ for $n = 0, 1, \ldots$. Find the first three approximants $x^{(1,2,3)}$ starting with $x^{(0)} = 1$.

   (c) $\langle 2 \rangle$ Find $x^{(\infty)}$ and by comparing with the exact solution, say where the iterative solution converges to the exact solution.

3. $\langle 12 \rangle$ Define $D_t(\omega) = \frac{2}{\pi}\frac{\sin^2 \frac{1}{2}\omega t}{\omega^2 t}$ and $U(t) = \int_{-\infty}^{\infty} d\omega\, D_t(\omega)$.

   (a) $\langle 2 \rangle$ As $t \to \infty$ what values does $D_t(\omega)$ tend to for $\omega = 0$ and $\omega \neq 0$?

   (b) $\langle 2 \rangle$ Find $U(t)$ in terms of $U(1)$

   (c) $\langle 5 \rangle$ Show $U(1) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx = 1$ by using the Cauchy residue theorem of complex contour integration.

   (d) $\langle 3 \rangle$ Express $\lim_{t\to\infty} D_t(\omega)$ in terms of familiar quantities.