Quantum Mechanics 2, Autumn 2011 CMI
Problem set 11
Due by beginning of class on Monday November 7, 2011
Scattering Theory

1. Use the partial wave expansion in terms of phase shifts to relate the total cross section σ for scattering by a spherically symmetric potential $V(r)$ to the imaginary part of the forward scattering amplitude $f(\theta = 0)$. Hint: To find $f(0)$ use $P_l(x) = \frac{1}{2^l l!} \frac{d(x^2-1)^l}{dx^l}$ where $x = \cos\theta$, work out the cases $l = 0, 1, 2$ and observe a pattern.

2. The differential cross section for Coulomb scattering of a projectile of charge q and incident angular wave number k by a fixed charge Q located at $r = 0$ is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mQq}{4\hbar^2 k^2}\right)^2 \frac{1}{\sin^4(\theta/2)}. \tag{1}$$

   (a) Check that the differential cross section has the correct dimensions.
   (b) Find the total cross section for Coulomb scattering. Does it matter whether the charges are of the same or opposite signs?
   (c) In a polar plot, roughly sketch the angular distribution of scattered particles (differential cross section). In which direction do most of the scattered particles go?
   (d) What is the classical limit of the differential cross section for Coulomb scattering?

3. The radial Schrodinger eigenvalue problem for a free particle in spherical coordinates (for $u(\rho) = rR(r)$ where $\rho = kr$) takes the form of the spherical Bessel equation

$$\left(-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2}\right) u_l(\rho) = u_l(\rho). \tag{2}$$

   We seek to build up the solutions for $l \geq 1$ using 'raising operators' applied to the orthogonal solutions for $l = 0$, namely $R_0(\rho) = j_0(\rho) = \sin\rho/\rho$ and $R_0(\rho) = n_0(\rho) = -\cos\rho/\rho$.

   (a) In what sense are $j_0(\rho)$ and $n_0(\rho)$ orthogonal, and why is this reasonable?
   (b) Suppose we define the 'lowering operator' $d_l = \frac{d}{d\rho} + \frac{l+1}{\rho}$. Find the raising operator $d_l^\dagger$.
   (c) Show that the spherical Bessel equation can be 'factorized' as $d_l d_l^\dagger u_l = u_l$.
   (d) Show also that $d_l^\dagger d_l = d_{l+1} d_{l+1}^\dagger$.
   (e) Use this to deduce that $\left(d_{l+1} d_{l+1}^\dagger\right)(d_l^\dagger u_l) = d_l^\dagger u_l$. What is the use of this result?
   (f) Suppose we normalize so that $d_l^\dagger u_l = u_{l+1}$, then show that $R_{l+1} = \left(-\frac{d}{d\rho} + \frac{l}{\rho}\right) R_l(\rho)$.
   (g) Further simplify this result to conclude that $R_l = (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l R_0(\rho)$.
   (h) Use this to find the spherical Bessel & Neumann functions $j_1(\rho), j_2(\rho), n_1(\rho), n_2(\rho)$.
   (i) Try to argue why they must have the asymptotic behaviors

$$j_l(\rho) \to \frac{\sin\left(\rho - \frac{l\pi}{2}\right)}{\rho} \quad \text{and} \quad n_l(\rho) \to -\frac{\cos\left(\rho - \frac{l\pi}{2}\right)}{\rho} \quad \text{as } \rho \to \infty. \tag{3}$$