Quantum Mechanics 2, Autumn 2011 CMI

Problem set 10

Due by beginning of class on Monday October 31, 2011

Scattering Theory

1. Find the S-matrix for scattering from an attractive δ well in one dimension, $H = \frac{p^2}{2m} - g\delta(x)$. You may use the known results for the transmitted and reflected amplitudes for the standard scattering problems.

2. Verify that the S-matrix for the delta potential well is unitary.

3. Find the pole(s) of the S-matrix in the complex k-plane and compare the energy $E = \hbar^2 k^2/2m$ at the pole(s) with the energies of the bound states in this potential.

4. Consider the 1d scattering problem for an asymptotically vanishing real potential $V(x)$ with asymptotic amplitudes

$$\psi(x) \to \begin{cases} Ae^{ikx} + Be^{-ikx} & as \;\; x \to -\infty \\ Ce^{ikx} + De^{-ikx} & as \;\; x \to +\infty \end{cases} \tag{1}$$

Show that the S-matrix is unitary, i.e.,

$$\left\langle \begin{pmatrix} A \\ D \end{pmatrix}, \begin{pmatrix} A' \\ D' \end{pmatrix} \right\rangle = \left\langle S \begin{pmatrix} A \\ D \end{pmatrix}, S \begin{pmatrix} A' \\ D' \end{pmatrix} \right\rangle \tag{2}$$

Hint: Consider the Wronskian $W(\psi_1^*(x), \psi_2(x))$ where ψ_1, ψ_2 are two scattering eigenstates with the same energy E

$$\psi_1(x) \to \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{as } \; x \to -\infty \\ Ce^{ikx} + De^{-ikx} & \text{as } \; x \to +\infty \end{cases}, \quad \psi_2(x) \to \begin{cases} A'e^{ikx} + B'e^{-ikx} & \text{as } \; x \to -\infty \\ C'e^{ikx} + D'e^{-ikx} & \text{as } \; x \to +\infty. \end{cases} \tag{3}$$

5. Consider scattering in 3d against a potential $V(r)$. Calculate the gradient of the scattered wave $\psi(\vec{r}) = f(\theta, \phi)\frac{e^{ikr}}{r}$ and find its leading behavior as $r \to \infty$.