**Quantum Mechanics 1, Spring 2011 CMI**
Problem set 7
Due by beginning of class on Monday March 7, 2011
Bra-ket, Hermiticity, uncertainty principle

1. Let $u, v$ be vectors and $A$ an operator on $\mathcal{H}$. Simplify (a) $\langle iAv|u\rangle$, (b) $\langle u|iAv\rangle$ and (c) $\langle iAv|iu\rangle$ for the standard $L^2$ inner product. In other words, pull out the $i$'s properly!

2. Based on the reality of average experimental measurements of observables, we defined a hermitian operator $A$ as one with real expectation values in every state $\psi$, i.e., ($\langle\psi|A\psi\rangle \in \mathbb{R}$). Say why this is the same as
$$\langle\psi|A\psi\rangle = \langle A\psi|\psi\rangle. \qquad (1)$$

3. A more conventional definition of hermiticity is that the matrix elements of $A$ satisfy
$$\langle u|Av\rangle = \langle Au|v\rangle \qquad (2)$$
for any pair of states $u, v$. Say why this is the same as $A_{uv} = (A_{vu})^*$.

4. Now suppose $A$ satisfies (1). We wish to show that it also satisfies (2). To show this, we put $\psi = u + v$ and $\psi = u + iv$ in (1) and add the two resulting equations. Show that this reduces to $A_{uv} = (A_{vu})^*$. Thus the reality of expectation values in all states implies that $A$ is hermitian in the conventional sense. The converse is much simpler.

5. Consider a particle in a (real) potential $V(x)$. Suppose $\psi(x)$ is a solution of the time-independent Schrödinger equation with (real) energy eigenvalue $E$. Find another wave function that has the same eigenvalue $E$. When are the two eigenfunctions the same?

6. Use the result of the previous problem to show that for any energy eigenvalue $E$, one can always find a corresponding real eigenfunction of the hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. This feature is because $H$ is not just hermitian but also real-symmetric.

7. If $P, Q$ are hermitian, what can you say about the commutator $[P, Q]$? Can $[P, Q]$ be an observable?

8. It is possible to show (using Cauchy-Schwarz) that for position and momentum $x$ and $p$,
$$(\Delta x)^2\,(\Delta p)^2 \geq -\frac{1}{4}\langle[x, p]\rangle_\psi^2 \qquad (3)$$
where $(\Delta x)^2 = \langle x^2\rangle_\psi - \langle x\rangle_\psi^2$ is the variance of $x$ in the state $\psi$ and similarly for $\Delta p$. Show that this reduces to the Heisenberg uncertainty principle.

9. Consider the momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ of a particle that is constrained to move in the interval $[-1, 1]$. Give a convenient choice of boundary condition for $\psi(\pm 1)$ that ensures that $\hat{p}$ is hermitian. Give the physical meaning of the boundary condition that you propose.