Quantum Mechanics 1, Spring 2011 CMI

Problem set 6

Due by the beginning of class on Friday February 18, 2011

Wave function, Schrodinger Equation, Fourier transforms

1. Consider a free non-relativistic particle of mass m. In the lecture we assumed the time evolution of each Fourier component of a matter wave $\psi(x,t)$ was given by $e^{i(kx-\omega(k)t)}$ corresponding to a right moving wave if $k, \omega(k)$ were of the same sign. We could equally well have considered the time evolution $e^{i(kx+\omega(k)t)}$. We do this here. Write down an expression for $\phi(x,t)$ for this 'left moving' wave packet. Assume that at $t = 0$, it has a shape $\tilde{\phi}(k)$ in k-space. $\langle 1.5 \rangle$

2. Derive the partial differential equation satisfied by this alternative wave packet $\phi(x,t)$. $\langle 4 \rangle$

3. How is this new PDE for $\phi(x,t)$ related to the Schrödinger equation? How is $\phi(x,t)$ related to the Schrödinger wave function $\psi(x,t)$? $\langle 1 \rangle$

4. Consider a quantum mechanical particle moving in a potential $V(x)$ in one dimension. Its state evolves according to the Schrodinger equation. Use the Schrödinger equation to calculate the time evolution of the mean momentum of the particle, $\frac{d\langle p \rangle}{dt}$ and express the answer in terms of the expectation values of other familiar quantities. Recall that $\langle 4 \rangle$

$$\langle p \rangle = \int dx\, \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi(x). \tag{1}$$

5. Suppose the wave function of a particle in one dimension is bounded and decays like $\psi(x) \sim \frac{1}{|x|^\alpha}$ as $|x| \to \pm\infty$ for some power $\alpha > 0$. How small can the power α be and still ensure that the wave function has a finite norm? $\langle 3 \rangle$

6. Evaluate the one-dimensional gaussian integral I_1 in closed form.

$$I_1 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \tag{2}$$

   Hint: Consider the Gaussian integral I_2, and evaluate it by transforming to polar coordinates on the x-y plane. How is I_2 related to I_1? $\langle 3 \rangle$

$$I_2 = \iint dx\, dy\, e^{-x^2-y^2} \tag{3}$$

7. Suppose $\tilde{f}(k)$ is the Fourier transform of $f(x)$,

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \tag{4}$$

   Find the Fourier transform $\tilde{g}(k)$ of the function $g(x) = x f(x)$. In other words, express $\tilde{g}(k)$ in terms of $\tilde{f}(k)$. $\langle 2.5 \rangle$

8. Multiplication by x in position space is represented by what operation in k-space? $\langle 1 \rangle$