Quantum Mechanics 1, Spring 2011 CMI

Problem set 4

Due by the beginning of class on Friday February 4, 2011

Waves, Photons and Bohr Atom

1. Suppose a source (possibly in a microwave oven) radiates electromagnetic waves at a power of 900 Watts in a collimated beam in the $\hat{x}$ direction. What is the force on the source? $\langle 2 \rangle$

2. How many photons from a 100 MHz beam of FM radio waves must an electron absorb before it has gained an energy of 10 eV? $\langle 1 \rangle$

3. Is the discreteness of the energy in an electromagnetic wave more easily detected for microwaves or X-rays. Why? $\langle 2 \rangle$

4. Consider the Bohr-Sommerfeld quantization condition for angular momentum in a circular electron orbit in the hydrogen atom. Express this condition in terms of the *action variable* $J = \oint p dq$. What are the appropriate coordinate and momentum q, p in this case? $\langle 1 \rangle$

5. Suppose a particle's trajectory in classical mechanics is a closed curve (the coordinates q^i are periodic in time with period τ). Show that the *action variable* $J = \sum_i \oint p_i \, dq^i$ that appears in the Bohr-Sommerfeld quantization rule is closely related to the action $S = \int L dt$ evaluated along one period of the trajectory. Here $L = T - V(q)$ is the Lagrangian for a non-relativistic particle of mass m. Find $J - S$ in terms of other familiar quantities. $\oint p \; dq$ denotes a line integral along a closed trajectory in configuration space. $\langle 4 \rangle$

6. Find the value of the speed of the electron in the most tightly bound Bohr orbit in the Hydrogen atom. Is it reasonable to treat the electron motion non-relativistically? Explain the occurrence of the speed of light in the the ionization energy of a hydrogen atom, the Rydberg energy $\mathbb{R} = \frac{1}{2} mc^2 \alpha^2$ where α is the so-called fine-structure constant. $\langle 3 \rangle$

7. Consider a wave packet moving in a medium with dispersion relation $\omega = \omega(k)$

$$\psi(x,t) = \int \frac{dk}{2\pi} \tilde{\psi}(k) e^{i(kx - \omega(k)t)} \tag{1}$$

   Suppose $\tilde{\psi}(k)$ is localized near a single wave number $k = k_0$ and falls to zero rapidly away from $k = k_0$ (e.g. the characteristic function of an interval $[k_0 - \delta k, k_0 + \delta k]$). We wish to find the location $x(t)$ of the wave packet at time t. To do so, we must find at what x the Fourier components constructively interfere. For most values of x at the given time t, the phase $i(kx - \omega(k)t)$ is significantly different for the various values of k in the interval of integration, i.e. the modes are out-of-phase and destructively interfere . Find the value of $x = x_0$ (at the given time) at which the modes are approximately in-phase and constructively interfere. $\langle 3 \rangle$

8. Use this condition of constructive interference to find the group velocity of the packet. $\langle 1 \rangle$

9. Define the width of an intensity distribution $I(x)$ as $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ where $\langle f(x) \rangle = \frac{\int I(x) \, f(x) \, dx}{\int I(x) \, dx}$. Find the widths σ_1, σ_2 of the following distributions. Which is narrower?

$$I_1(x) = \cos^2 x \quad \text{and} \quad I_2(x) = \cos^4 x, \qquad x \in [-\pi/2, \pi/2] \tag{2}$$

   Hints: You may use $\int_{-\pi/2}^{\pi/2} x^2 \cos^2 x \, dx = \frac{1}{24}(\pi^3 - 6\pi)$ and $\int_{-\pi/2}^{\pi/2} x^2 \cos^4 x \, dx = \frac{1}{64}(2\pi^3 - 15\pi)$. The average value of $\cos^4 x$ is $3/8$. What is the average value of $\cos^2 x$? $\langle 3 \rangle$