Quantum Mechanics 1, Spring 2011 CMI

Problem set 2

Due by the beginning of class on Friday January 21, 2011

Classical motion for zero angular momentum in a $-\frac{g}{r}$ potential

Let us try to model a hydrogen atom as a simple classical mechanical system. It is assumed to have an infinitely heavy point-like nucleus that exerts a radially inward directed force of magnitude $\frac{g}{r^2}$ on a point-like electron of mass m. Suppose further that an experiment told us that the angular momentum of the electron in the hydrogen atom is zero.

So consider a particle of mass m moving in three dimensional space under the influence of a central potential $V(r) = -\frac{g}{r}$. Recall that in spherical polar coordinates

$$z = r\cos\theta, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \tag{1}$$

the Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{g}{r}. \tag{2}$$

Let us denote the Hamiltonian for the corresponding free particle ($g = 0$) by H_0.

1. Show that the square of the angular momentum vector (total angular momentum) is precisely the angular part of the hamiltonian H

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \tag{3}$$

   Hint: Use the vector identity $(\vec{r}\times\vec{p})^2 = r^2p^2 - (\vec{r}\cdot\vec{p})^2$. ⟨3⟩

2. Give an example of a *state* with zero angular momentum $\vec{L} = 0$ (located at a finite distance from the origin and with finite energy $E < 0$) for such a particle. ⟨2⟩

3. Write the Hamiltonian and Hamilton's equations in spherical coordinates for a particle with zero angular momentum in the above potential. ⟨2⟩

4. What are the cyclic coordinates and conserved momenta? ⟨1⟩

5. Reduce hamilton's equations to a single second order differential equation for $r(t)$. Find this equation and write down the initial conditions corresponding to the above zero angular momentum state. ⟨1⟩

6. Is the above differential equation linear or non-linear? ⟨1⟩

7. Qualitatively explain the motion of the particle that follows from the above equation of motion and initial condition. Give a rough plot of $r(t)$. ⟨4⟩

8. Is the energy conserved during the motion? ⟨1⟩

9. Where is the particle located after a sufficiently long time? ⟨1⟩

10. Classical electromagnetism predicts that a charged particle emits radiation when accelerated. If the above particle was charged, would it radiate when trying to follow the above trajectory? How would this affect the motion? ⟨2⟩

11. How does the Earth escape the fate that is predicted for the above particle? ⟨2⟩