Quantum Mechanics 1, Spring 2011 CMI

Problem set 13

Due by beginning of class on Monday April 18, 2011

Angular momentum

1. Using the expressions for the differential operators L_z, L_x, L_y in spherical coordinates

$$L_z = -i\hbar \frac{\partial}{\partial \phi}, \quad L_x = i\hbar \left(\sin\phi \frac{\partial}{\partial \theta} + \cot\theta \cos\phi \frac{\partial}{\partial \phi} \right), \quad L_y = i\hbar \left(-\cos\phi \frac{\partial}{\partial \theta} + \cot\theta \sin\phi \frac{\partial}{\partial \phi} \right) \quad (1)$$

show that the square of angular momentum $L^2 = L_x^2 + L_y^2 + L_z^2$ takes the simple form

$$L^2 \psi = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right]. \quad (2)$$

Hint: For brevity, denote partial derivatives by subscripts.

2. We saw that L^2 and L_z are compatible observables, $[L^2, L_z] = 0$. Here we begin the task of finding common eigenstates for L^2 and L_z. Define the raising and lowering operators

$$L_+ = L_x + iL_y, \quad L_- = L_x - iL_y, \quad L_\pm = L_x \pm iL_y. \quad (3)$$

   (a) Show that

$$[L^2, L_\pm] = 0 \qquad \text{and} \qquad [L_z, L_\pm] = \pm\hbar L_\pm. \quad (4)$$

   (b) Now suppose ψ is a simultaneous eigenfunction of L^2 and L_z

$$L^2 \psi = \lambda \psi, \quad L_z \psi = \hbar m \, \psi. \quad (5)$$

   What are the physical dimensions of λ and m? Is m the mass of the particle?

   (c) Can λ be less than zero? Why? Give a state ψ for which λ is zero.

   (d) Using the commutation relations (4), show that $L_\pm \psi$ are also eigenfunctions of L^2. What are the corresponding eigenvalues?

   (e) Show that $L_\pm \psi$ are eigenfunctions of L_z. What are the corresponding eigenvalues?

   (f) Suppose $L^2 \psi = \lambda \psi$ and $L_z \psi = \hbar m \psi$. How large and how small can m be compared to λ? Hint: Use positivity of $L^2 = L_x^2 + L_y^2 + L_z^2$.