Quantum Mechanics 1, Spring 2011 CMI
Problem set 11
Due by beginning of class on Monday April 4, 2011
Heisenberg equation of motion, Scattering in one dimension

1. Show that an operator A commutes with any power of itself A^n for $n = 0, 1, 2, \cdots$

$$[A, A^n] = 0. \tag{1}$$

2. Show that the commutator with p of any function of x is $[p, f(x)] = -i\hbar f'(x)$.

3. Consider a free particle in one dimension $H = \frac{p^2}{2m}$.

   (a) Find Heisenberg's equations of motion for the position and momentum operators $x_h(t)$ and $p_h(t)$ (you can drop the subscripts h).

   (b) Solve the above Heisenberg equations of motion in terms of the initial operators $q_h(0)$ and $p_h(0)$.

   (c) Compute the commutator of the position operators at different times $[q(0), q(t)]$.

   (d) Applying the uncertainty relation $(\Delta A)^2(\Delta B)^2 \geq -\frac{1}{4}\langle [A, B]\rangle^2$ in a Heisenberg state ψ_h, find a lower bound for the uncertainty product $\Delta q(0)\Delta q(t)$.

   (e) Suppose we prepare a free particle in an initial state with a certain spread in positions $\Delta q(0)$. Based on the above calculations, what can you predict about the minimal spread in positions for the state as time t grows?

4. Consider a particle in a real potential $V(x)$ that vanishes as $x \to \pm\infty$. The Schrodinger eigenvalue problem is

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi \tag{2}$$

   (a) Suppose ψ and ϕ are both eigenfunctions with the same energy eigenvalue. Show that the Wronskian $W(\psi, \phi)(x) = \psi\phi' - \psi'\phi$ is independent of x.

   (b) Suppose ψ is a scattering energy eigenstate with asymptotic behavior

$$\psi(x) \to a e^{-ikx} \quad \text{as } x \to -\infty, \quad \text{and} \quad \psi(x) \to e^{-ikx} + b e^{ikx} \quad \text{as } x \to \infty. \tag{3}$$

   What is the energy eigenvalue of $\psi(x)$? What is the physical interpretation of such a scattering state wave function?

   (c) Is $\psi^*(x)$ also an energy eigenstate, and if so with what energy?

   (d) Find the Wronskian $W(\psi, \psi^*)(x)$ as $x \to \pm\infty$. How must these asymptotic values of the Wronskian be related?

   (e) Use the above to find a relation between a and b. Give a physical interpretation of the relation.