Quantum Mechanics 1, Spring 2011 CMI

Problem set 1

Due by 5pm Thursday January 13, 2011

Hamiltonian and Lagrangian formulation for a particle in a central potential

Consider a particle of mass m moving in three dimensional space under the influence of a potential $V(r)$ that depends only on the distance r from a given point (origin). Its energy in Cartesian coordinates is

$$E = T + V = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(\sqrt{x^2 + y^2 + z^2}). \tag{1}$$

Define spherical polar coordinates by choosing an axis (say the z-axis) and defining the angle made by the radius vector with it as θ. The projection of the radius vector on the orthogonal $x - y$ plane then makes an angle ϕ, say with the x-axis. Then

$$z = r\cos\theta, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi. \tag{2}$$

1. How many degrees of freedom does the particle have? $\langle 1 \rangle$
2. Express the components of velocity $(\dot{x}, \dot{y}, \dot{z})$ in spherical coordinates. $\langle 3 \rangle$
3. Find the Lagrangian in spherical coordinates $L(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}) = T - V$. This involves some calculation but be assured that most of the terms cancel out in the end. $\langle 4 \rangle$
4. Find the conjugate momenta p_r, p_θ, p_ϕ. $\langle 3 \rangle$
5. Express the energy in terms of r, θ, ϕ and their conjugate momenta. $\langle 2 \rangle$
6. Obtain the Hamiltonian from the Legendre transform of the Lagrangian and check that it is the same as the energy. $\langle 3 \rangle$

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \text{ext}_{\dot{r},\dot{\theta},\dot{\phi}}\left(p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right) \tag{3}$$

7. What are the cyclic coordinates in the Hamiltonian and the corresponding conserved momenta? $\langle 2 \rangle$
8. Is $L_x = yp_z - zp_y$ a conserved quantity? Hint: You could calculate $\dot{L}_x$ directly. But instead try to use the answer to the previous question and do no further calculation. Give an argument that uses a different choice of initial axis while setting up spherical polar coordinates. $\langle 3 \rangle$
9. What is the *state* of minimum energy for the above particle if $V(r) = -\frac{1}{r}$. $\langle 3 \rangle$
10. Find Hamilton's equations in spherical coordinates. $\langle 3 \rangle$
11. Find Lagrange's equations in spherical coordinates. $\langle 3 \rangle$