Problems on formulation and basic mathematical tools of quantum mechanics

National Workshop on Formulation and Approximation Methods of Quantum Mechanics

22-24 Oct, 2018 at Government Brennen College, Thalassery, Kerala

Govind S. Krishnaswami, Chennai Mathematical Institute

govind@cmi.ac.in, http://www.cmi.ac.in./~govind

Classical Mechanics

1. How many degrees of freedom do the following systems have? (a) One point particle moving on a circle (b) Two point particles moving on a circle (c) Three point particles moving in a lecture hall (d) an irregularly shaped stone moving in the air above the playground.

2. Find Hamilton's equations for a particle in a double well potential with Hamiltonian

$$H = \frac{p^2}{2m} + g(x^2 - a^2)^2, \quad m, g, a > 0. \tag{1}$$

Check that they reduce to Newton's second order equation.

3. If one postulates the anti-symmetric Poisson brackets $\{x, p\} = -\{p, x\} = 1$ etc., check that the equations of motion $\dot{f}(x, p) = \{f(x, p), H\}$ reduce to Hamilton's equations for $f = x$ and $f = p$ for the above particle.

4. Find all static solutions to Hamilton's equations and classify them as stable/unstable to small perturbations.

5. Draw a phase portrait for the above particle in a double well potential.

Quantum Mechanics

6. Consider the Pauli matrices in the standard basis. Now $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Find $\sigma_2^\dagger$. Show that σ_2 is hermitian and unitary.

7. What is σ_2^2? What is the characteristic equation for σ_2?

8. Find a simple formula (in terms of trigonometric functions) for $e^{i\theta\sigma_2/2}$ by summing the exponential series.

9. What are the eigenvalues of σ_2? Find the corresponding eigenvectors v_+ and v_-.

10. Show that v_+ and v_- satisfy the completeness relation $v_+ v_+^\dagger + v_- v_-^\dagger = |v_+\rangle\langle v_+| + |v_-\rangle\langle v_-| = I_{2\times 2}$.

11. Show that the eigenvalue λ of an operator A may be interpreted as the expectation value of A in the corresponding eigenstate.

12. Show that the commutator of two hermitian operators is anti-hermitian.

13. Suppose a is any operator on a Hilbert space and let $H_1 = a^\dagger a$ and $H_2 = a a^\dagger$. Check that H_1 and H_2 are hermitian. Show that expectation values of H_1 and H_2 in all (non-zero) states are non-negative. We say that $aa^\dagger$ and $a^\dagger a$ are positive (more precisely non-negative) operators. Hint: recall the definition of the norm of a vector.

14. Argue that the expectation value of kinetic energy $T = p^2/2m$ is nonnegative in any state. Also argue that the expectation value of energy $T + \frac{1}{2}m\omega^2 x^2$ of a simple harmonic oscillator is positive in any state.

15. Consider two finite dimensional matrices A, B. What is $\text{tr}\,[A, B]$?

16. Consider the Heisenberg commutation relations $[x, p] = i\hbar I$. Comment on the result of calculating the trace of either side.

17. Recall that $\int_{-\infty}^{\infty} e^{iky}\, dy = 2\pi\delta(k)$. Evaluate the integral $\int_{-\infty}^{\infty} y\, e^{iky}\, dy$.

18. Find the matrix element of the position operator between momentum eigenstates $\langle k|x|k'\rangle$. Hint: $\langle x|k\rangle = e^{ikx}$ and use the completeness relation.

19. Find the matrix element of the momentum operator p between position eigenstates $\langle x|p|y\rangle$.

20. Given two observables (hermitian operators) A, B with $[A, B] = iC$ one may show the uncertainty inequality $(\Delta A)_\psi^2 (\Delta B)_\psi^2 \geq \frac{1}{4}\langle C\rangle_\psi^2$ for any (unit norm) state ψ. Say how $(\Delta A)_\psi$ is defined. Here $\langle C\rangle_\psi$ is the expectation value of C in the state ψ. Consider the angular momentum observables that satisfy $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$. Apply the uncertainty inequality to $A = L_x$ and $B = L_y$ and comment on the result. Is there a state where the product of uncertainties can vanish? Contrast this with the case $A = x, B = p$.

21. Consider the Schrödinger equation for a particle in a 1d real potential $V(x)$:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x). \tag{2}$$

Find the equation satisfied by the complex conjugate wave function.

22. Derive a local conservation law for probability in 1D: $\frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0$ where $P = |\psi(x,t)|^2$. What is the probability current $j(x,t)$?

23. Show that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$.

24. Show that the Gaussian wave packet $\psi(x) = Ae^{-\frac{x^2}{4a^2}}$ has unit norm if we take $A = \frac{1}{\sqrt{a}(2\pi)^{1/4}}$. What is the mean position and mean momentum of a particle in this state?