Particle Physics, Autumn 2014 CMI

Problem set 8

Due at the beginning of lecture on Tuesday Jan 27, 2015

Equivalence of adjoint and spin-1 representations of SU(2) Lie algebra

1. ⟨**11**⟩ We are now familiar with two 3d unitary representations of the SU(2) Lie algebra. The adjoint representation and the angular momentum one representation from quantum mechanics (coming from $L_3|m\rangle = m|m\rangle$ and $L_\pm = \sqrt{2 - m(m \pm 1)}|m \pm 1\rangle$ in units where $\hbar = 1$)

$$I_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad I_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

$$L_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

We will show that these two are equivalent representations by finding a unitary change of basis that transforms them into each other, i.e., $U^\dagger I_\alpha U = L_\alpha$ for each $\alpha = 1, 2, 3$. U must diagonalize I_3 to L_3, however, it is not unique, each column can be multiplied by a phase. These phases have to be chosen so that the same U transforms $I_{1,2}$ into $L_{1,2}$. [Note: Unlike SU(2), SU(3) has representations of the same dimension which are inequivalent.]

   (a) ⟨**6**⟩ Find the unitary transformations U that diagonalize I_3, i.e., $U^\dagger I_3 U = L_3$. Recall that the columns of U are the unit norm eigenvectors of I_3 in the same order as the eigenvalues along the diagonal of L_3. So find these eigenvectors, allowing for phases ϕ_1, ϕ_2, ϕ_3 and specify U.

   (b) ⟨**3**⟩ Find the conditions on the phases to ensure that $U^\dagger I_1 U = L_1$ and $U^\dagger I_2 U = L_2$.

   (c) ⟨**2**⟩ Find the general solution to the conditions on phases. Pick a convenient set of phases to arrive at a specific explicit unitary equivalence U between the adjoint and 'spin 1' representations.