Particle Physics, Autumn 2014 CMI
Problem set 13
Due on Wednesday March 4, 2015
$[E, B]$, Euler-Lagrange equations, complex scalar coupled to EM, ϵ_{ij},

1. ⟨**6**⟩ Show that the equal-time commutator between quantized electric and magnetic fields is

$$[E_i(\mathbf{r}), B_j(\mathbf{r}')] = -i\hbar c\, \epsilon_{ijk} \frac{\partial}{\partial r_k} \delta^3(\mathbf{r} - \mathbf{r}'). \tag{1}$$

2. ⟨**2**⟩ The Lagrangian for a complex scalar field with a self-interaction V is $\mathcal{L} = |\partial\phi|^2 - m^2|\phi|^2 - V(\phi^*, \phi)$. What is the equation of motion for ϕ?

3. ⟨**3**⟩ The Lagrangian for a real scalar field with self interaction V is $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)$. What is the equation of motion for ϕ? Explain the reason for the factor $\frac{1}{2}$ difference between real and complex scalar Lagrangians.

4. ⟨**4**⟩ Show that the Euler-Lagrange equations following from $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu$ are the inhomogeneous Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$. Here $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

5. ⟨**4**⟩ A complex scalar field coupled to an electromagnetic gauge potential has Lagrangian density $\mathcal{L} = (\partial_\mu + ieA_\mu)\phi^* \, (\partial^\mu - ieA^\mu)\phi - V(\phi^*\phi)$. Expand out the terms in this Lagrangian and try to give a physical interpretation to the various terms.

6. ⟨**3**⟩ Under a change of basis S_{ia}, the totally anti-symmetric ϵ tensor ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$) in 2d transforms to $\epsilon'_{ij} = S_{ia}S_{jb}\epsilon_{ab}$. Show explicitly that $\epsilon'_{ij} = (\det S)\epsilon_{ij}$. In particular, how does ϵ_{ij} transform under an SU(2) change of basis S? Note: $\epsilon'_{ijk} = (\det S)\,\epsilon_{ijk}$ holds in 3 dimensions etc.