Particle Physics, Autumn 2014 CMI
Problem set 12
Due at the beginning of lecture on Tuesday Feb 24, 2015
Minimal coupling, Zero point energy of EM field

1. ⟨**13**⟩ Recall the Newton-Lorentz classical equation for a charged particle moving in an EM field $m\ddot{\mathbf{r}} = \vec{F} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$ where the fields may be expressed in terms of potentials $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$. We will work out Hamilton's equations $\dot{r}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial r_j}$ and show that they reduce to the Lorentz force law if the hamiltonian is chosen as $H = \frac{1}{2m}\left(p - \frac{eA}{c}\right)^2 + e\phi$. We work in classical mechanics so $A_i p_j = A_j p_i$ etc.

   (a) ⟨**4**⟩ Use Hamilton's equations to show that

   $$m\dot{r}_j = p_j - \frac{e}{c}A_j \quad \text{and} \quad -\dot{p}_j = e\frac{\partial\phi}{\partial r_j} + \frac{e^2}{mc^2}A_i\frac{\partial A_i}{\partial r_j} - \frac{e}{mc}p_i\frac{\partial A_i}{\partial r_j} \tag{1}$$

   (b) ⟨**4**⟩ Combine these and simplify to obtain

   $$m\ddot{r}_j = eE_j + e\frac{v_i}{c}\frac{\partial A_i}{\partial r_j} - \frac{e}{c}(v\cdot\nabla)A_j. \tag{2}$$

   Here $v_i = \frac{dr_i}{dt}$. Hint: $A_j = A_j(\mathbf{r}(t), t)$ is evaluated at the location of the particle.

   (c) ⟨**5**⟩ Simplify the vector identity

   $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \tag{3}$$

   in the case where $\mathbf{B} = \vec{v}(t)$ is the velocity vector and $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$ is the vector potential field. Use it to obtain the Newton-Lorentz equation $m\ddot{r}_j = eE_j + \frac{e}{c}(v\times B)_j$.

2. ⟨**9**⟩ Fourier integral for zero point energy of photon field in radiation gauge. The prescription for converting Fourier sums to Fourier integrals is $\frac{1}{V}\sum_{\mathbf{k}} \to \int\frac{d^3k}{(2\pi)^3}$, $V\delta_{\mathbf{k}\mathbf{k}'} \to (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$, while the creation and destruction operators are $a(\mathbf{k},\lambda) = \sqrt{V}a_{\mathbf{k},\lambda}$, $a^\dagger(\mathbf{k},\lambda) = \sqrt{V}a^\dagger_{\mathbf{k},\lambda}$

   (a) ⟨**2**⟩ What are the dimensions of $a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k},\lambda}, a(\mathbf{k},\lambda)$ and $a^\dagger(\mathbf{k},\lambda)$?

   (b) ⟨**1**⟩ What is $[a(\mathbf{k},\lambda), a^\dagger(\mathbf{k}',\lambda')]$?

   (c) ⟨**3**⟩ We have already shown that

   $$H = \frac{1}{2}\int(\mathbf{E}^2 + \mathbf{B}^2)d^3\mathbf{r} = \frac{1}{2}\sum_{\mathbf{k},\lambda}\hbar\omega_{\mathbf{k}}(a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} + a_{\mathbf{k}\lambda}a^\dagger_{\mathbf{k}\lambda}) \tag{4}$$

   Convert this to a Fourier integral and show that

   $$H = \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\hbar\omega_{\mathbf{k}}a^\dagger(\mathbf{k},\lambda)a(\mathbf{k},\lambda) + (2\pi)^3\delta^3(0)\int\frac{d^3k}{(2\pi)^3}\hbar c|\mathbf{k}|. \tag{5}$$

   The second term is the zero point energy.

   (d) ⟨**2**⟩ Say why we should interpret the momentum space delta function $(2\pi)^3\delta^3(0)$ as the volume of three dimensional position space, i.e., V (which has become infinite).

   (e) ⟨**1**⟩ Is the zero point energy finite or infinite when $V\to\infty$?