Particle Physics, Autumn 2014 CMI
Problem set 10
Due at the beginning of lecture on Tuesday Feb 10, 2015
Lorentz invariant volume element and Radiation gauge

1. ⟨**4**⟩ It is often necessary to integrate quantities (f below) over all possible on-shell 4-momenta of a particle of mass m. These momenta lie on the upper sheet of a two sheeted hyperboloid in Minkowski space. Show that

$$\int \frac{d^4 p}{(2\pi)^4} f(p^0, \mathbf{p}) 2\pi \delta(p^2 - m^2) \theta(p^0 > 0) = \int \frac{d^3 p}{2E_{\mathbf{p}}(2\pi)^3} f(E_p, \mathbf{p}). \tag{1}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ is the energy of the particle and θ is the Heaviside step function.

2. ⟨**4**⟩ The group of gauge transformations parametrized by $\theta(\mathbf{x}, t)$ acts on the space of gauge potentials via

$$\mathbf{A}' = \mathbf{A} + \nabla\theta, \quad \phi' = \phi - \frac{1}{c}\frac{\partial \theta}{\partial t}. \tag{2}$$

Gauge potentials that differ by a gauge transformation represent the same electric and magnetic fields and are said to lie on a common orbit of the gauge group. A gauge choice is a choice of orbit representative. Coulomb gauge is defined by the condition $\nabla \cdot \mathbf{A} = 0$. Given a vector potential $\mathbf{A}'$ find the gauge transformation θ that transforms it to a vector potential $\mathbf{A}$ in Coulomb gauge. i.e., find the equation that θ must satisfy and an integral expression for the solution. Hint: Recall the formula for the electrostatic potential due to a given charge distribution.

3. ⟨**9**⟩ Suppose we use radiation gauge $\phi = 0, \nabla \cdot \mathbf{A} = 0$ and expand $\mathbf{A}$ in Fourier modes

$$\mathbf{A}(\mathbf{r}, t) = \frac{c}{\sqrt{V}} \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

c is the speed of light and radiation fills a large cavity of volume V, with $\int d^3\mathbf{r}\ e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = V\delta_{\mathbf{k}\mathbf{k}'}$. Moreover, reality of $\mathbf{A}(\mathbf{r}, t)$ implies $\mathbf{A}_{-\mathbf{k}} = \mathbf{A}^*_{\mathbf{k}}$.

   (a) ⟨**3**⟩ Show that the electric ($-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$) and magnetic ($\nabla \times \mathbf{A}$) fields are

$$\mathbf{E} = -\frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \dot{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \quad \text{and} \quad \mathbf{B} = \frac{ic}{\sqrt{V}} \sum_{\mathbf{k}} (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

   Hint: You need to use a vector identity to get the expression for the magnetic field.

   (b) ⟨**6**⟩ Show that the electric and magnetic energies are

$$\frac{1}{2}\int \mathbf{E}^2 d^3\mathbf{r} = \frac{1}{2}\sum_{\mathbf{k}} |\dot{\mathbf{A}}_{\mathbf{k}}|^2 \quad \text{and} \quad \frac{1}{2}\int \mathbf{B}^2 d^3\mathbf{r} = \frac{c^2}{2}\sum_{\mathbf{k}} |\mathbf{k} \times \mathbf{A}_{\mathbf{k}}|^2. \tag{5}$$