Nonlinear Dynamics, Spring 2020 CMI
Problem set 6
Due at the beginning of lecture on Wednesday Mar 18, 2020
2d vector fields and their linearization

1. $\langle \mathbf{12} \rangle$ The simplest 2d autonomous nonlinear systems $\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r})$ are those that consist of uncoupled 1d systems. Consider for instance such a system with 'minimal' nonlinearity:

$$\dot{x} = -x + x^2 = x(x-1) \quad \text{and} \quad \dot{y} = -y. \tag{1}$$

   (a) $\langle \mathbf{1} \rangle$ Find the fixed points.
   (b) $\langle \mathbf{2} \rangle$ Find the nullclines.
   (c) $\langle \mathbf{3} \rangle$ Sketch a phase portrait using the nullclines and use arrows to indicate the direction of flow along typical trajectories.
   (d) $\langle \mathbf{3} \rangle$ Find the linearization of the vector field (Jacobian) at the fixed points.
   (e) $\langle \mathbf{3} \rangle$ What does the linearization predict about the nature of the fixed points and how does this compare with the phase portrait of the nonlinear system?

2. $\langle \mathbf{14} \rangle$ Consider the system $\dot{\theta} = 1$ and $\dot{r} = -ar^n$ in plane polar coordinates $r = (x^2 + y^2)^{1/2} \geq 0$, $\theta = \arctan(y/x)$ for $n = 1, 2, 3, \ldots$ and say $a > 0$.

   (a) $\langle \mathbf{3} \rangle$ Plot a rough phase portrait (for $n = 1$ and $n = 2$) indicating the direction of flow on typical trajectories.
   (b) $\langle \mathbf{5} \rangle$ Express these equations in Cartesian coordinates, i.e., find $\dot{x}$ and $\dot{y}$.
   (c) $\langle \mathbf{3} \rangle$ Compute the 2×2 Jacobian matrix at the fixed point(s) in Cartesian variables.
   (d) $\langle \mathbf{3} \rangle$ What does the linearization (in Cartesian coordinates) say about the fixed point(s) for $n = 1, 2, 3, \ldots$ and how does this compare with the behavior of the nonlinear system?