Nonlinear Dynamics, Spring 2020 CMI
Problem set 5
Due at the beginning of lecture on Monday Feb 17, 2020
Linear vector fields on the plane

1. ⟨**5**⟩ Suppose $\mathbf{w}_\pm$ are eigenvectors corresponding to the eigenvalues $\lambda_\pm$ of a real 2×2 matrix A. Assuming the eigenvalues are distinct, argue that the eigenvectors can be taken either both having real components or to be complex conjugates of eachother: $\mathbf{w}_- = \mathbf{w}_+^*$.

2. ⟨**15**⟩ Consider the harmonic oscillator $\dot{x} = p/m$ and $\dot{p} = -kx$ with $\omega = \sqrt{k/m}$. Take $\mathbf{r} = (x, p)$ so that $\dot{\mathbf{r}} = A\mathbf{r}$.

   (a) ⟨**5**⟩ Find the coefficient matrix A and its eigenvalues $\lambda_\pm$. Comment on the implications of the eigenvalues of A for the nature of the fixed point (and its stability) at $(x_* = 0, p_* = 0)$.

   (b) ⟨**5**⟩ Find the eigenvectors $\mathbf{w}_\pm$ of A corresponding to the eigenvalues $\lambda_\pm$ and show that they can be taken to be complex conjugates. For definiteness, try to ensure that the first component of both eigenvectors is 1 (choice of normalization).

   (c) ⟨**5**⟩ Use these eigenvalues and eigenvectors to write the general solution of the equations for a harmonic oscillator in the form (for some complex number r_+)

$$x(t) = 2(\Re r_+)\cos\omega t - 2(\Im r_+)\sin\omega t \quad \text{and} \quad p(t) = 2m\omega\left[-\Re r_+ \sin\omega t - \Im r_+ \cos\omega t\right]. \quad (1)$$

3. ⟨**5**⟩ Give an example of a 2×2 real matrix with only one linearly independent eigenvector. Find its eigenvalues and the lone independent eigenvector.