Mathematical Physics 1: Linear Algebra, CMI

Problem set 9

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Due at the beginning of class (9:15am) on Fri, Sep 4.

Eigenvalue problem associated to a matrix

Given a matrix H, the associated eigenvalue problem is $Hx = \lambda x$. The problem is to find all complex numbers (*eigenvalues*) λ for which there is a non-zero vector x satisfying this equation. In quantum mechanics, H is the energy operator. The possible eigenvalues are the possible energies of the system. The corresponding eigenvector x is the wavefunction of the state with energy λ. As an example consider $H = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. The eigenvalue problem is the system of equations $(H - \lambda I)x = 0$ where I is the 2×2 identity matrix.

1. Find the condition on λ for $H - \lambda I$ to have a non-trivial kernel. $< 2 >$

2. The above condition must be a quadratic equation $\lambda^2 + b\lambda + c = 0$, called the characteristic equation. Find b, c. $< 1 >$

3. Solve this condition and find the allowed eigenvalues λ. (Hint: there should be two $\lambda_1 < \lambda_2$) $< 2 >$

4. For each eigenvalue λ_1, λ_2, find the corresponding eigenvectors, column vectors u_1, u_2 (Hint: Use Gaussian elimination to solve $(H - \lambda_1 I)u_1 = 0$. Check that the answer satisfies $Hu_1 = \lambda_1 u_1$ for instance). $< 3 >$

5. Show that the eigenvectors corresponding to eigenvalue λ_1 *span* a vector space. What is the dimension of the eigen-space corresponding to the eigenvalue λ_1? $< 2 >$

6. Find the determinant of H and compare it with the product of eigenvalues as well as with the coefficient c determined above. $< 2 >$

7. The trace of H, $\text{tr}\, H$ is defined as the sum of its diagonal elements. Find $\text{tr}\, H$ and compare it to the sum of eigenvalues as well as to the coefficient $-b$ found earlier. $< 2 >$

8. Using the eigenvalues, calculate the matrix product $(H - \lambda_1)(H - \lambda_2) = H^2 - (\lambda_1 + \lambda_2)H + \lambda_1 \lambda_2$. $< 1 >$

9. Using the previous result, find H^9 without multiplying H explicitly 9 times. $< 1 >$

10. Explain based on H, why you could have expected the particular numerical value obtained for the smaller eigenvalue λ_1. (Hint: what is the meaning of the eigenvalue problem for $\lambda = \lambda_1$?) $< 1 >$

11. Calculate the expected value of the energy in the state u_2, which is defined as $E_2 = \frac{u_2^T H u_2}{u_2^T u_2}$. Compare it with λ_2. $< 1 >$

12. Calculate the dot product of eigenvectors $u_1 \cdot u_2 = u_1^T u_2$. Comment on the geometrical meaning of the answer. $< 2 >$