Mathematical Physics 1: Linear Algebra, CMI

Problem set 6

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Due at the beginning of class on Tuesday, August 25.

General solution of $Ax = b$ and orthogonality of subspaces

Consider the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix}$

1. Between which vector spaces is A a linear transformation? Give the dimensions of the domain and target. $< 1 >$

2. Using row elimination bring A to row echelon (upper triangular) form. What is its rank and pivots? $< 2 >$

3. Write the equation for the kernel $Ax = 0$ in echelon form. Which are the pivot variables and free variables? $< 1 >$

4. Find the special solutions which form a basis for $N(A)$ by prescribing convenient values (0's and 1's as described in the lecture) to the free variables. Verify that these basis vectors are orthogonal to the row space of A. $< 2 >$

5. Assemble the special solutions as the columns of a matrix N, the null space matrix. Do you notice any relation between N and the reduced row echelon form of A? $< 2 >$

6. Express $N(A)$ as the span of its basis vectors. What is the dimension of the null space and does it agree with the rank-nullity theorem? $< 2 >$

7. Find the null space of A^T by the same procedure as above. (Start with A^T and bring it to row echelon form, etc.) $< 2 >$

8. Is the column vector $\tilde{b} = (1 \quad 6 \quad 0)^T$ orthogonal to $N(A^T)$? What does this imply for the consistency of $Ax = \tilde{b}$? $< 1 >$

9. Is the column vector $b = (1 \quad 6 \quad 7)^T$ orthogonal to $N(A^T)$? What does this imply for the consistency of $Ax = b$? $< 1 >$

10. Find *the* particular solution to $Ax = b$ for $b = (1 \quad 6 \quad 7)^T$ that corresponds to the special choice of setting all free variables to zero. $< 1 >$

11. Using the above particular solution and the basis for the null space write a formula for the general solution to $Ax = b$ for the above b. How many parameters parameterize the general solution? $< 2 >$

12. Suppose in the system of equations $Ax = b$ we know that $A_{m \times n}$ has rank $r = n - 1$. What is $\dim N(A)$? Suppose further we know that $m = n + 1$. Then what is the dimension of the space of $\tilde{b}'s$ for which there is no solution to $Ax = \tilde{b}$? Do such $\tilde{b}$'s form a vector space? By *dimension* we mean the number of real parameters needed to specify the $\tilde{b}$'s for which there is no solution. $< 2 >$

13. Give an example of a matrix (or class of matrices) for which the column space $C(A)$ is orthogonal to the null space $N(A)$ (you may identify isomorphic vector spaces). $< 1 >$