Mathematical Physics 1: Linear Algebra, CMI

Problem set 11

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Due at the beginning of class on Mon, Sep 14.

Diagonalization, Eigenvalues and Eigenvectors

1. The matrix $L = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ acting on $\mathbf{C}^3$ represents (up to a constant factor) a component of angular momentum in quantum mechanics. Is L hermitian or symmetric?

2. What do you expect about the angles between the eigenvectors of L and why?

3. Find the characteristic equation for L. The spectrum of a matrix is the set of eigenvalues. Find the spectrum of L. Name the eigenvalues appropriately using the labels $\lambda_0, \lambda_\pm$ with $\lambda_+ > 0$. Assemble the eigenvalues in a diagonal matrix $\Lambda = diag(\lambda_-, \lambda_0, \lambda_+)$.

4. Find the eigenspaces (expressed as span of some vectors) of the eigenvalues.

5. What are the dimensions of the eigenspaces of L? Is L deficient?

6. Assemble the eigenvectors (normalized to 1) as the columns of a matrix $U = (u_-, u_0, u_+)$. What sort of matrix is U? Why? (Note: if an eigenspace is 1-dimensional, you need to include only one eigenvector from it in U, not the most general vector in the eigenspace.)

7. What is the expected (numerical) value of L in the state u_0, i.e., $u_0^\dagger L u_0$? Give the answer using the eigenvalue problem, without explicit matrix multiplication.

8. What is U^{-1}? (Hint: This does not need a long calculation.)

9. Evaluate the similarity transformation $U^{-1}LU$ and compare it with the matrix Λ.

10. What are the matrix elements of L in the basis specified by the columns of U?

11. For the matrix $N = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ find the eigenvalues and their algebraic and geometric multiplicities. Is N deficient in eigenvectors?

12. Consider the Pauli matrices as linear transformations from $\mathbf{C}^2 \to \mathbf{C}^2$. In the standard cartesian o.n. basis, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What is their commutator?

13. Find the eigenvalues and corresponding linearly independent (normalized to 1) eigenvectors of σ_2.

14. Using the eigenvectors, find the unitary transformation U that diagonalizes σ_2. Check that it does the job i.e. $U^\dagger \sigma_2 U = \Lambda$.

15. Find the matrix representation of σ_3 in the eigenbasis of σ_2.

16. Are σ_2 and σ_3 simultaneously diagonalizable? Why?