Problems in basic linear algebra

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Govind S. Krishnaswami, Chennai Mathematical Institute
http://www.cmi.ac.in/~govind/teaching, email: govind@cmi.ac.in

1. **Matrix multiplication and Pauli Matrices**: Pauli matrices are the 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

They are important in quantum mechanics and group theory. Here $i = \sqrt{-1}$ is the imaginary unit with $i^2 = -1$. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 identity matrix.

(a) For $1 \leq i, j \leq 3$, ϵ_{ijk} is the Levi-Civita symbol (epsilon tensor). $\epsilon_{123} = 1$ and it is antisymmetric under the interchange of any two neighbouring indices, such as $\epsilon_{ijk} = -\epsilon_{jik}$. Find ϵ_{ijk} for all possible values of $1 \leq i, j, k \leq 3$.

(b) Using these results, verify that the products of the Pauli matrices can be summarized as

$$\sigma_a \sigma_b = \delta_{ab} I + i \epsilon_{abc} \sigma_c, \qquad \text{where} \quad a, b = 1, 2, 3. \tag{2}$$

The repeated index c is summed from 1 to 3.

(c) The commutator of a pair of matrices measures to what extent $AB \neq BA$. More precisely, $[A, B] = AB - BA$. Using the above results, find $[\sigma_1, \sigma_2], [\sigma_2, \sigma_3], [\sigma_3, \sigma_1]$ and express the answers in terms of the Pauli matrices. The final answer should fit in one line.

2. **Matrix of discretized derivative**: Newton's equation $\ddot{x} = f$ can be written as a matrix equation when discretized. Here you will do this for the simpler problem of the first derivative. Given the position of a particle $x(t)$, find its (approximate) velocity. We are provided the positions of the particle $x_k \equiv x(t_k)$ at equally spaced times $t_1, t_2, \cdots, t_n$, with $t_{i+1} - t_i = \Delta$.

(a) Assemble the positions of the particle in a column vector with n-components X and display it.

(b) The velocity is $\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta) - x(t)}{\Delta}$. Define the approximate velocity $\dot{x}_k$ at any time t_k as the difference quotient with $\Delta = 1$. Write a formula for $\dot{x}_k$. You may assume that the particle returned to its original position at the end of the journey $x(t_{n+1}) = x(t_1)$.

(c) List out $\dot{x}_k$ for $k = 1, 2, 3, n-1, n$.
The approximate velocities are assembled in a column vector $V = \begin{pmatrix} \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_n \end{pmatrix}^t$

(d) Find the matrix D, which when applied to the column of positions, produces the column of approximate velocities $V = DX$.

(e) Write out the matrix D_n for the case $n = 4$ explicitly.

(f) What vector space do V and X live in?

(g) Is D_4 upper triangular? Is D_4 symmetric?

(h) What is the rank of D_4?

(i) What is the determinant of D_4? Is it invertible?

(j) Find a column vector annihilated by D_4. If there is a non-zero vector in the kernel of D_4, find it, otherwise explain why there isn't one.

(k) What sort of physical motion does the above-discovered vector in the kernel represent?

(l) Can you guess all vectors in the kernel of D_n and their physical meaning?

3. **Linear transformation**:

(a) Consider the reflection R of any vector in $\mathbf{R}^2$ about the x-axis. Write in components what R does to a general vector.

(b) Is R a linear transformation? Why?

(c) If it is a linear transformation, find the matrix representation of the reflection R in the standard cartesian basis for $\mathbf{R}^2$.

(d) The matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$ is a toy version of the annihilation operator in quantum mechanics. Find

i. its rank,

ii. its determinant,

iii. all vectors it annihilates,

iv. a 3-component column vector b for which $Ax = b$ has no solution.

4. **Projections, Orthogonal matrices**:

(a) Let $A = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Is A anti-symmetric? Why?

(b) Find A^n for all $n = 0, 1, 2, \cdots$. (Hint: the answer is very simple, A^n is periodic in n.)

(c) Define the matrix exponential for any real number x, as the matrix $e^{Ax} = \sum_{n=0}^{\infty} \frac{A^n x^n}{n!}$. Obtain a formula for e^{Ax} as a linear combination $e^{Ax} = f(x)I + g(x)A$. Find $f(x), g(x)$.

(d) Using the above-obtained formula, find whether e^{Ax} is an orthogonal matrix.

5. **Eigenvalue problem associated to a matrix**:

Given a matrix H, the associated eigenvalue problem is $Hx = \lambda x$. The problem is to find all complex numbers (*eigenvalues*) λ for which there is a non-zero vector x satisfying this equation. In quantum mechanics, H is the energy operator. The possible eigenvalues are the possible energies of the system. The corresponding eigenvector x is the wavefunction of the state with energy λ. As an example consider $H = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. The eigenvalue problem is the system of equations $(H - \lambda I)x = 0$ where I is the 2×2 identity matrix.

(a) Find the condition on λ for $H - \lambda I$ to have a non-trivial kernel.

(b) The above condition must be a quadratic equation $\lambda^2 + b\lambda + c = 0$, called the characteristic equation. Find b, c.

(c) Solve this condition and find the allowed eigenvalues λ. (Hint: there should be two $\lambda_1 < \lambda_2$)

(d) For each eigenvalue λ_1, λ_2, find the corresponding eigenvectors, column vectors u_1, u_2 (Hint: Use Gaussian elimination to solve $(H - \lambda_1 I)u_1 = 0$. Check that the answer satisfies $Hu_1 = \lambda_1 u_1$ for instance).

(e) Show that the eigenvectors corresponding to eigenvalue λ_1 *span* a vector space. What is the dimension of the eigen-space corresponding to the eigenvalue λ_1?

(f) Find the determinant of H and compare it with the product of eigenvalues as well as with the coefficient c determined above.

(g) The trace of H, $\text{tr}\, H$ is defined as the sum of its diagonal elements. Find $\text{tr}\, H$ and compare it to the sum of eigenvalues as well as to the coefficient $-b$ found earlier.

(h) Using the eigenvalues, calculate the matrix product $(H - \lambda_1)(H - \lambda_2) = H^2 - (\lambda_1 + \lambda_2)H + \lambda_1\lambda_2$.

(i) Using the previous result, find H^9 without multiplying H explicitly 9 times.

(j) Explain based on H, why you could have expected the particular numerical value obtained for the smaller eigenvalue λ_1. (Hint: what is the meaning of the eigenvalue problem for $\lambda = \lambda_1$?)

(k) Calculate the expected value of the energy in the state u_2, which is defined as $E_2 = \frac{u_2^T H u_2}{u_2^T u_2}$. Compare it with λ_2.

(l) Calculate the dot product of eigenvectors $u_1^T u_2$. Comment on its geometrical meaning.

6. **Determinants, Eigenvalues and Eigenvectors**:

(a) Consider the change of coordinates from spherical polar to cartesian coordinates in three dimensional Euclidean space $(r, \theta, \phi) \mapsto (x, y, z)$ where $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$

$$z = r\cos\theta, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi. \tag{3}$$

Draw a diagram indicating $(x, y, z), (r, \theta, \phi)$ for a point in the interior of the first octant.

(b) Find the Jacobian matrix J for the transformation of coordinates.

(c) The change in volume element is $dxdydz = dr\, d\theta\, d\phi\, \det J$. Find $\det J$ using cofactors.

(d) Consider the matrix $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ representing a linear transformation $Q : R^2 \to R^2$. Describe what it does to vectors.

(e) Does Q have any real eigenvectors? Why? (Answer without explicit calculations)

(f) Find the eigenvalues of Q.

(g) Now regard Q as a linear transformation $Q : C^2 \to C^2$. Find eigenvectors (with norm $= 1$) corresponding to the above eigenvalues.

7. **Diagonalization, Eigenvalues and Eigenvectors**:

(a) The matrix $L = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ acting on $\mathbf{C}^3$ represents (up to a constant factor) a component of angular momentum in quantum mechanics. Is L hermitian or symmetric?

(b) What do you expect about the angles between the eigenvectors of L and why?

(c) Find the characteristic equation for L. The spectrum of a matrix is the set of eigenvalues. Find the spectrum of L. Name the eigenvalues appropriately using the labels $\lambda_0, \lambda_\pm$ with $\lambda_+ > 0$. Assemble the eigenvalues in a diagonal matrix $\Lambda = diag(\lambda_-, \lambda_0, \lambda_+)$.

(d) Find the eigenspaces (expressed as span of some vectors) of the eigenvalues.

(e) What are the dimensions of the eigenspaces of L? Is L deficient?

(f) Assemble the eigenvectors (normalized to 1) as the columns of a matrix $U = (u_-, u_0, u_+)$. What sort of matrix is U? Why? (Note: if an eigenspace is 1-dimensional, you need to include only one eigenvector from it in U, not the most general vector in the eigenspace.)

(g) What is the expected (numerical) value of L in the state u_0, i.e., $u_0^\dagger L u_0$? Give the answer using the eigenvalue problem, without explicit matrix multiplication.

(h) What is U^{-1}? (Hint: This does not need a long calculation.)

(i) Evaluate the similarity transformation $U^{-1}LU$ and compare it with the matrix Λ.

(j) What are the matrix elements of L in the basis specified by the columns of U?

(k) For the matrix $N = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ find the eigenvalues and their algebraic and geometric multiplicities. Is N deficient in eigenvectors?

(l) Consider the Pauli matrices as linear transformations from $\mathbf{C}^2 \to \mathbf{C}^2$. In the standard cartesian o.n. basis, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What is their commutator?

(m) Find the eigenvalues and corresponding linearly independent (normalized to 1) eigenvectors of σ_2.

(n) Using the eigenvectors, find the unitary transformation U that diagonalizes σ_2. Check that it does the job i.e. $U^\dagger \sigma_2 U = \Lambda$.

(o) Find the matrix representation of σ_3 in the eigenbasis of σ_2.

(p) Are σ_2 and σ_3 simultaneously diagonalizable? Why?

8. **Principal Axis Transformation**:

(a) Consider the quadratic curve E in the $x-y$ plane defined by the equation $2x^2 + 3xy - yx + 2y^2 = 1$. Write this equation as a matrix equation and identify the real symmetric matrix A whose quadratic form is involved.

(b) Plot the curve E roughly on the $x-y$ plane. (Find a few points on E and join the dots, the figure must show the major and minor axes roughly)

(c) Do the $x - y$ coordinate axes point along the principal axes of E? Why or why not?

(d) What is the condition for the position vector of a point P to point in the same direction as the normal?

(e) Find the principal axes of E by interpreting it as an eigenvalue problem.

(f) Find the lengths of the semi-major and semi-minor axes.

(g) Indicate the principal axes and their lengths in a figure.

(h) Find the particular principal axis transformation Q for the above quadratic curve satisfying $\det Q = +1$ What sort of transformation is Q, describe its action on the coordinate axes? (Hint: This and the next question involve choices of order!)

(i) Find a different principal axis transformation Q' with $\det Q' = -1$. Describe the action of Q' on the coordinate axes.

(j) Explain the need for reflections in the passage to principal axes and in the choice of eigenvalue matrix Λ in the above example.

(k) Is A a positive definite matrix? Why?

(l) Find e^A for the above matrix A using the principal axis transformation.