Flow of Fluids

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- The illustrations in these slides are scanned from books mentioned or downloaded from the internet and thanks are due to the creators/owners of the images.
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Water, water, everywhere . . . (S T Coleridge, Rime of the Ancient Mariner)

- Whether we do physics, chemistry, biology, computation, mathematics, engineering or the humanities, we are likely to encounter fluids and be fascinated and challenged by their flows.
- Fluid flows are all around us: the air through our nostrils, tea stirred in a cup, water down a river, charged particles in the ionosphere etc.
- Let us take a few minutes to brainstorm and write down terms and phenomena that come to our mind when we think of fluid flows.

Terms that come to mind in connection with fluids

- flow, waves, ripples, sound, wake,
- water, air, hydrodynamics, aerodynamics, lift, drag, flight.
- velocity, density, pressure, viscosity, streamlines,
- laminar, turbulent, chaotic
- vortex, bubble, drop
- convection, clouds, plumes, hydrological cycle
- weather, climate,
- rain, flood, hurricane, tornado, cyclone, typhoon, tsunami,
- shock, sonic boom, compressible, incompressible,
- surface, surface tension, splash,
- solar flares, aurorae, plasmas

Clouds from plumes

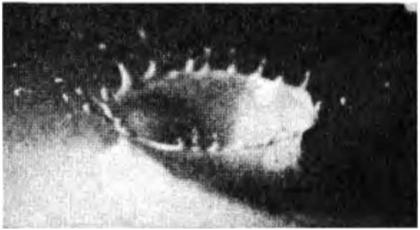
Plume of ash and gas from Mt. Etna, Sicily, 26 Oct, 2013. NASA.

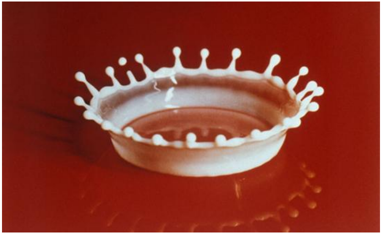
Water water every where . . .

- Some of the best scientists have worked on fluid mechanics: I Newton, D Bernoulli, L Euler, J L Lagrange, Lord Kelvin, H Helmholtz, C L Navier, G G Stokes, N Y Zhukovsky, M W Kutta, O Reynolds, L Prandtl, T von Karman, G I Taylor, J Leray, L F Richardson, A N Kolmogorov, L Onsager, R P Feynman, L D Landau, S Chandrasekhar, O Ladyzhenskaya, etc.
- Fluid dynamics finds application in numerous areas: flight of airplanes and birds, weather prediction, blood flow in the heart and blood vessels, waves on the beach, ocean currents and tsunamis, controlled nuclear fusion in a tokamak, jet engines in rockets, motion of charged particles in the solar corona and astrophysical jets, accretion disks around active galactic nuclei, formation of clouds, melting of glaciers, climate change, sea level rise, traffic flow, building pumps and dams etc.
- Fluid motion can be appealing to the senses and also present us with mysteries and challenges.
- Fluid flows can range from regular and predictable (laminar) to seemingly disorganized and chaotic (turbulent) while displaying remarkable patterns.

Splashes from a drop of milk

- Arthur Worthington (1879) and Harold Edgerton (1935) took photos of splashes of milk. One sees a remarkable undulating corona in such a splash.
- Symmetry breaking - initially we have circular symmetry in the liquid annulus, but as the splash develops, segmentation occurs and spikes emerge at regular intervals reducing the symmetry to a discrete one.
- How did Worthington take such a photograph?





Worthington's and Edgerton's milk splashes

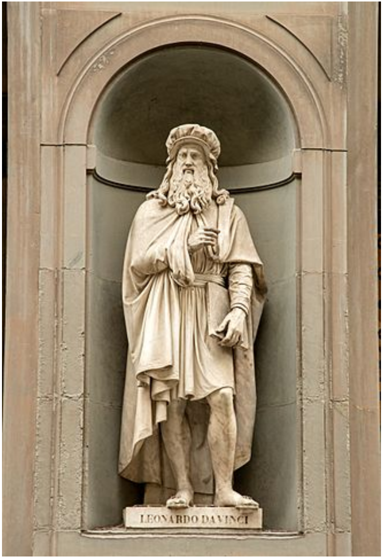
Mathematical modelling of fluid phenomena

- “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” – article published in 1960 by the physicist Eugene Wigner.
- His concluding paragraph: The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.
- Relationships between physical quantities in a flow are fruitfully expressed using differential equations. Before discussing these equations, we will introduce fluid phenomena through pictures and mention some of the physical concepts and approximations developed to understand them.

Language of fluid mechanics: pictures and calculus

- Leonardo da Vinci (1452-1519) wanted to understand the flow of water. He had neither the laws of Newton nor the tools of calculus at his disposal. Nevertheless he made much progress by observing flows and trying to understand and use them. His notebook Codex Leicester contains detailed accounts of his observations, discoveries, questions and reflections on the subject.
- It was not until the time of I Newton (1687), D Bernoulli (1738) and L Euler (1757) that our understanding of the laws of fluid mechanics began to take shape and mathematical modelling became possible.
- Mathematical modelling of natural/behavioral phenomena is not always very successful. Sometimes the phenomena do not match the predictions of the models we propose. Sometimes we do not even know the laws to formulate appropriate models.
- We believe we know the physical laws governing fluid motion. However, despite much progress since the time of Euler, it is still a challenge to predict and understand many features of the flows around us.

Leonardo da Vinci



Continuum, Fluid element, Fields

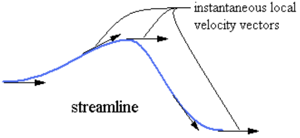
- In fluid mechanics we are not interested in microscopic positions and velocities of individual molecules. Focus instead on macroscopic fluid variables like velocity, pressure, density, energy and temperature that we can assign to a fluid element by averaging over it.
- By a fluid element, we mean a sufficiently large collection of molecules so that concepts such as 'volume occupied' make sense and yet small by macroscopic standards so that the velocity, density, pressure etc. are roughly constant over its extent. E.g.: divide a container with 10^{23} molecules into 10000 cells, each containing 10^{19} molecules.
- Thus, we model a fluid as a continuum system with an essentially infinite number of degrees of freedom. A point particle has 3 translational degrees of freedom. On the other hand, to specify the pattern of a flow, we must specify the velocity at each point!
- Fluid description applies to phenomena on length-scale $\gg$ mean free path. On shorter length-scales, fluid description breaks down, but Boltzmann's kinetic theory of molecules applies.

Concept of a field: a gift from fluid flow

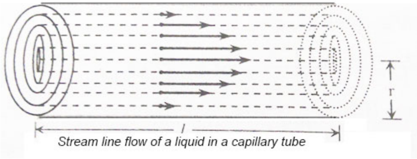
- The concept of a point particle is familiar and of enormous utility.
- We imagine a particle to be somewhere at any given time. By contrast, a field is everywhere at any given instant!
- Fluid and solid mechanics are perhaps the first places where the concept of a field emerged in a concrete manner.
- At all points of a fluid we have its density. It could of course vary from point to point $\rho(\mathbf{r})$. It could also vary with time: $\rho(\mathbf{r},t)$ is a dynamical field.
- Similarly, we have the pressure and velocity fields $p(\mathbf{r},t), \mathbf{v}(\mathbf{r},t)$. Unlike ρ and p which are scalars, $\mathbf{v}$ is a vector. At each point $\mathbf{r}$ it is represented by a little arrow that conveys the magnitude and direction of velocity.
- Fields also arose elsewhere: the gravitational field of Issac Newton and the electric and magnetic fields introduced by Michael Faraday. However, these fields are somewhat harder to grasp. They were introduced to explain the transmission of gravitational, electric and magnetic forces between masses, charged particles and magnets.

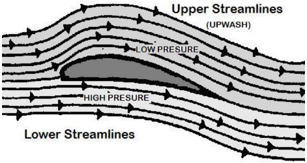
Flow visualization: Streamlines

- Streamlines encode the instantaneous velocity pattern. They are curves that are everywhere tangent to $\mathbf{v}$.
- If $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r})$ is time-independent everywhere, then the flow is steady and the streamlines are frozen. In unsteady flow, the stream lines continuously deform. Streamlines at a given time cannot intersect.



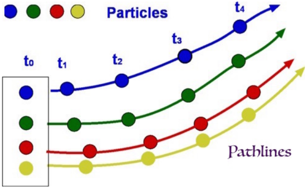
- A flow that is regular is called laminar. This happens in slow steady pipe flow, where streamlines are parallel. Another example is given in this movie of water flowing from a nozzle.





Flow visualization: Path-lines

- In practice, how do we observe a flow pattern?
- Leonardo suspended fine sawdust in water and observed the motion of the saw dust (which reflects light) as it was carried by the flow.
- This leads to the concept of path-lines.
- Path-lines are trajectories of individual fluid 'particles' (e.g. speck of dust stuck to fluid). At a point P on a path-line, it is tangent to $\mathbf{v}(P)$ at the time the particle passed through P. Pathlines can (self)intersect at $t_1 \neq t_2$.



Flow visualization: streak-lines

- Another approach is to continuously introduce a dye into the flow at some point and watch the pattern it creates.

- Streak-line: Dye is continuously injected into a flow at a fixed point P. Dye particle sticks to the first fluid particle it encounters and flows with it. Resulting high-lighted curve is the streak-line through P. So at a given time of observation t_{obs}, a streak-line is the locus of all current locations of particles that passed through P at some time $t \leq t_{\text{obs}}$ in the past.

- Video of numerical simulation of streaklines in cigarette plume.

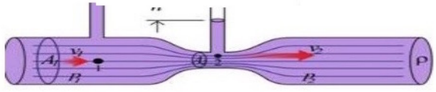
- Streamlines, path-lines and streak-lines all coincide for steady flow, but not for unsteady flow.

Bernoulli's principle

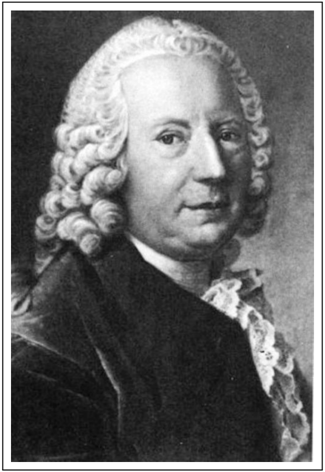
- Among the earliest quantitative observations about fluid flows is Bernoulli's principle: the pressure drops where a flow speeds up.
- In its simplest form, it applies to steady flow of a fluid of uniform density ρ and says that

$$\mathscr{B} = \frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho} + gz$$

  is constant along streamlines. Here g is the acceleration due to gravity and z the vertical height on the streamline.
- For roughly horizontal flow, pressure is lower where velocity is higher.
- Pressure drops as flow speeds up at constrictions in a pipe.
- Try to separate two sheets of paper by blowing air between them!



- $A_2 < A_1$; $V_2 > V_1$
- According to Bernoulli's Law, pressure at A_2 is lower.

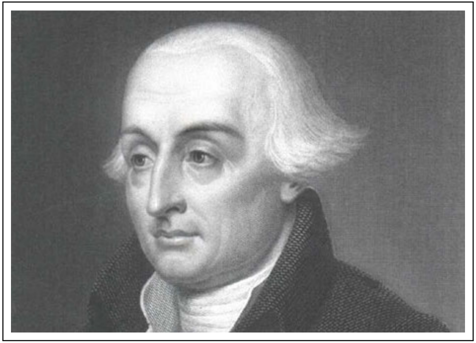


Daniel Bernoulli

Eulerian and Lagrangian viewpoints

- In the Eulerian description, we are interested in the time development of fluid variables at a given point of observation $\vec{r} = (x, y, z)$. Interesting if we want to know how density changes, say, above my head. However, different fluid particles will arrive at the point $\vec{r}$ as time elapses.
- It is also of interest to know how the corresponding fluid variables evolve, not at a fixed location but for a fixed fluid element, as in a Lagrangian description.
- This is especially important since Newton's second law applies directly to fluid particles, not to the point of observation!

Leonhard Euler and Joseph Louis Lagrange



Leonhard Euler (left) and Joseph Louis Lagrange (right).

Conservation of mass

- There are two primary laws of fluid motion.
- The conservation of mass states the obvious: the mass of a fluid element remains constant as the element moves around. The same collection of molecules reside in the element but the shape and size of the element can change with time.
- Said differently, the rate of increase in mass of fluid in a fixed volume must be due to the influx of material across its boundary.
- If the volume of a fluid element changes with time, we say the fluid is compressible. Typical flows in water are incompressible, while high speed flows in air tend to be compressible.
- To formulate mass conservation via an equation, we need to use the concept of a material derivative: it measures how the density $\rho(\mathbf{r}, t)$ of a fluid element changes as it moves around.

Material derivative measures rate of change along flow

- Change in density of a fluid element in time dt as it moves from $\mathbf{r}$ to $\mathbf{r}+d\mathbf{r}$ is

$$d\rho = \rho(\mathbf{r}+d\mathbf{r}, t+dt) - \rho(\mathbf{r}, t) \approx \frac{\partial \rho}{\partial t} dt + d\mathbf{r}\cdot\nabla\rho. \tag{1}$$

- Divide by dt, let $dt \to 0$ and use $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ to get instantaneous rate of change of density of a fluid element located at $\mathbf{r}$ at time t:

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho. \tag{2}$$

- $D\rho/Dt$ measures rate of change of density of a fluid element as it moves around. Material derivative of any quantity (scalar or vector) s in a flow field $\mathbf{v}$ is defined as $\frac{Ds}{Dt} = \partial_t s + \mathbf{v}\cdot\nabla s$.
- Material derivative of velocity $\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$ gives the instantaneous acceleration of a fluid element with velocity $\mathbf{v}$ located at $\mathbf{r}$ at time t.
- As a 1^{st} order differential operator it satisfies Leibnitz' product rule

$$\frac{D(fg)}{Dt} = f\frac{Dg}{Dt} + g\frac{Df}{Dt} \quad \text{and} \quad \frac{D(\rho\mathbf{v})}{Dt} = \rho\frac{D\mathbf{v}}{Dt} + \mathbf{v}\frac{D\rho}{Dt}. \tag{3}$$

Continuity equation and incompressibility

- Rate of increase of mass in a fixed vol V is equal to the influx of mass. Now, $\rho \mathbf{v} \cdot \hat{n}\, dS$ is the mass of fluid leaving a volume V through a surface element dS per unit time. Here $\hat{n}$ is the outward pointing normal. Thus,

$$\frac{d}{dt}\int_V \rho\, d\mathbf{r} = -\int_{\partial V} \rho \mathbf{v}\cdot\hat{n}\, dS = -\int_V \nabla\cdot(\rho\mathbf{v})\, d\mathbf{r} \quad \Rightarrow \int_V [\rho_t + \nabla\cdot(\rho\mathbf{v})]\, d\mathbf{r} = 0.$$

- As V is arbitrary, we get continuity equation for local mass conservation:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \text{or} \qquad \partial_t\rho + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0. \tag{4}$$

- In terms of material derivative, $\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$.
- Flow is incompressible if $\frac{D\rho}{Dt} = 0$: density of a fluid element is constant. Since mass of a fluid element is constant, incompressible flow preserves volume of fluid element.
- Alternatively incompressible means $\nabla\cdot\mathbf{v} = 0$, i.e., $\mathbf{v}$ is divergence-free or solenoidal. $\nabla\cdot\mathbf{v} = \lim_{V,\delta t\to 0} \frac{1}{\delta t}\frac{\delta V}{V}$ measures fractional rate of change of volume of a small fluid element.
- Most important incompressible flow is constant ρ in space and time.

Sound speed, Mach number

- Incompressibility is a property of the flow and not just the fluid! For instance, air can support both compressible and incompressible flows.
- Flow may be approximated as incompressible in regions where flow speed is small (subsonic) compared to local sound speed $c_s = \sqrt{\frac{\partial p}{\partial \rho}} \sim \sqrt{\gamma p/\rho}$ for adiabatic flow of an ideal gas with $\gamma = c_p/c_v$. Sound is a disturbance by which density variations propagate in a fluid.
- Compressibility $\beta = \frac{\partial \rho}{\partial p}$ measures increase in density with pressure. Incompressible fluid has $\beta = 0$, so $c^2 = 1/\beta = \infty$. An approximately incompressible flow is one with very large sound speed ($c_s \gg |\mathbf{v}|$).
- Common flows in water are incompressible. So study of incompressible flow is called hydrodynamics. High speed flows in air/gases tend to be compressible. Compressible flow is called aerodynamics/ gas dynamics.
- Incompressible hydrodynamics may be derived from compressible gas dynamic equations in the limit of small Mach number $M = |\mathbf{v}|/c_s \ll 1$.
- When $M \gg 1$ we have super-sonic flow and phenomena like shocks.

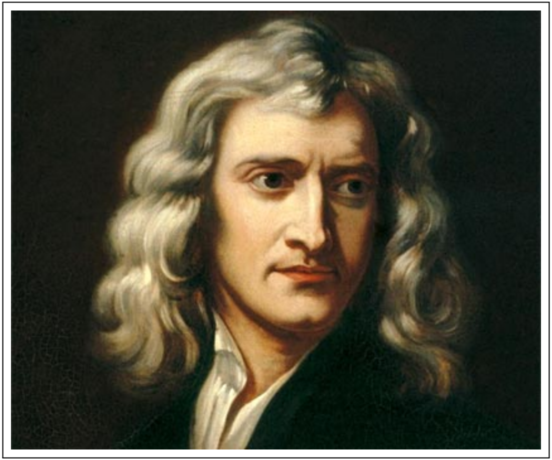
Newton's second law for a fluid

- Newton's second law of motion for a particle says $m\mathbf{a} = \mathbf{F}$. Its mass times its acceleration is equal to the force acting on it. In other words, forces cause the velocity to change.
- The precise mathematical form of Newton's 2nd law for a fluid (ignoring viscous dissipation) was derived by Leonhard Euler (1757).
- What does Newton's law say for a small fluid element of volume δV? If ρ is the density of fluid then its mass is $\rho\delta V$. The acceleration is the rate of change of its velocity along the flow: $\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}$.
- To apply Newton's law to a fluid element we need to know the forces that act on it.
- There are three main forces: gravity, pressure and frictional/viscous forces exerted by neighboring elements. Thus:

$$\rho(\delta V)\frac{D\mathbf{v}}{Dt} = \rho(\delta V)\mathbf{g} + \text{pressure and viscous forces.}$$

• Here $\mathbf{g}$ is the acceleration due to gravity, 9.8 m/s^2 acting downwards.

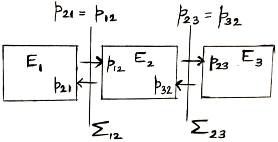
Isaac Newton



Isaac Newton

What are pressure and viscous forces?

- Consider a small element E_2 of fluid and its neighbouring elements, E_1 to the left and E_3 to the right. The elements are separated by imaginary surfaces/membranes Σ_{ij}: $E_1 \Sigma_{12} E_2 \Sigma_{23} E_3$.



- The molecules in E_1 collide with those of E_2 in the vicinity of the surface Σ_{12}. The normal component of this surface force (per unit area) is called the pressure p_{12} due to E_1 on E_2.
- Pressure provides a nice illustration of Newton 3rd law: the force exerted by E_1 on E_2 is equal and opposite to the force E_2 exerts on E_1. Thus the pressure $p_{12} = p_{21}$ does not depend on which element one focuses on.
- On the other hand, the normal surface force p_{32} exerted by E_3 on E_2 need not be exactly opposite to that exerted by E_1 on E_2. Such a pressure imbalance $(p_{12} = p_{21} \neq p_{32} = p_{23})$ or pressure gradient can cause the fluid element E_2 to accelerate and generate a flow.
- Viscous forces are also surface forces, they are the tangential components of the forces between elements.

Newton's 2^{nd} law for fluid element: Inviscid Euler equation

- Consider a fluid element of volume δV. Mass $\times$ acceleration is $\rho(\delta V)\frac{D\mathbf{v}}{Dt}$.
- Force on fluid element includes 'body force' like gravity $\mathbf{F} = \rho(\delta V)\mathbf{g}$.
- Also have surface force on a volume element, due to pressure exerted on it by neighbouring elements

$$\mathbf{F}_{\text{surface}} = -\int_{\partial V} p\hat{n}\,dS = -\int_V \nabla p\,dV; \quad \text{if} \quad V = \delta V \quad \text{then} \quad \mathbf{F}_{\text{surf}} \approx -\nabla p(\delta V).$$

- Newton's 2^{nd} law then gives the celebrated (inviscid) Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}; \qquad \mathbf{v}\cdot\nabla\mathbf{v} \to \text{'advection term'} \tag{5}$$

- Continuity ($\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$) & Euler are 1^{st} order in time: to solve initial value problem, must specify $\rho(\mathbf{r}, t=0)$ and $\mathbf{v}(\mathbf{r}, t=0)$.
- **Boundary conditions:** Euler equation is 1st order in space derivatives; impose BC on $\mathbf{v}$, not $\partial_i\mathbf{v}$. On solid boundaries normal component of velocity vanishes $\mathbf{v}\cdot\hat{n} = 0$. As $|\mathbf{r}| \to \infty$, typically $\mathbf{v} \to 0$ and $\rho \to \rho_0$.

Consequence of Euler equation: Sound waves (Video)

- Sound waves are excitations of the ρ or p fields. Arise in compressible flows, where regions of compression and rarefaction can form.
- Notice first that a fluid at rest ($\mathbf{v} = 0$) with *constant* pressure and density ($p = p_0$, $\rho = \rho_0$) is a static solution to the continuity and Euler equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{and} \quad \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p. \tag{6}$$

- Now suppose the stationary fluid suffers a small disturbance resulting in small variations $\delta \mathbf{v}, \delta p$ and $\delta \rho$ in velocity, pressure and density

$$\mathbf{v} = 0 + \mathbf{v}_1(\mathbf{r},t), \quad \rho = \rho_0 + \rho_1(\mathbf{r},t) \quad \text{and} \quad p = p_0 + p_1(\mathbf{r},t). \tag{7}$$

  What can the perturbations $\mathbf{v}_1(\mathbf{r},t), p_1(\mathbf{r},t)$ and $\rho_1(\mathbf{r},t)$ be? They must be such that $\mathbf{v}, p$ and ρ satisfy the continuity and Euler equations with $\mathbf{v}_1, p_1, \rho_1$ treated to linear order (as they are assumed small).

- It is found empirically that the small pressure and density variations are proportional i.e., $p_1 = c^2 \rho_1$. We will derive the simplest equation for sound waves by linearizing the continuity and Euler eqns around the static solution. It will be possible to interpret c as the speed of sound.

Sound waves in static fluid with constant p_0, ρ_0

- Ignoring products of small quantities $\mathbf{v}_1, p_1$ and ρ_1, the continuity equation $\partial_t(\rho_0+\rho_1)+\nabla\cdot((\rho_0+\rho_1)\mathbf{v}_1)=0$ becomes $\partial_t\rho_1+\rho_0\nabla\cdot\mathbf{v}_1=0$.
- Similarly, the Euler equation $(\rho_0+\rho_1)(\partial_t\mathbf{v}_1+\mathbf{v}_1\cdot\nabla\mathbf{v}_1)=-\nabla(p_0+p_1)$ becomes $\rho_0\partial_t\mathbf{v}_1=-\nabla p_1$ upon ignoring products of small quantities.
- Now we assume pressure variations are linear in density variations ($p_1=c^2\rho_1$) and take a divergence to get $\rho_0\partial_t(\nabla\cdot\mathbf{v}_1)=-c^2\nabla^2\rho_1$.
- Eliminating $\nabla\cdot\mathbf{v}_1$ using continuity eqn we get the wave equation for density variations $\partial_t^2\rho_1=c^2\nabla^2\rho_1$.
- Why is c called the sound speed? Notice that any function of $\xi=x-ct$ solves the 1D wave equation: $\partial_t^2\rho_1=c^2\partial_x^2\rho_1$ for $\rho_1(x,t)=f(x-ct)$

$$\partial_t\rho_1=-cf', \quad \partial_t^2\rho_1=c^2f'' \quad \text{while} \quad \partial_x\rho_1=f' \quad \text{and} \quad \partial_x^2\rho_1=f''. \tag{8}$$

  $f(x-ct)$ is a *traveling wave* that retains its shape as it travels at speed c to the right. Plot $f(x-ct)$ vs x at $t=0$ and $t=1$ for $f(\xi)=e^{-\xi^2}$ and $c=1$.
- For incompressible flow ($\rho=\rho_0,\rho_1=0$) $c^2=\frac{p_1}{\rho_1}=\frac{\delta p}{\delta\rho}\to\infty$ as the density variation is vanishingly small even for large pressure variations.

Including viscosity: Navier-Stokes equations

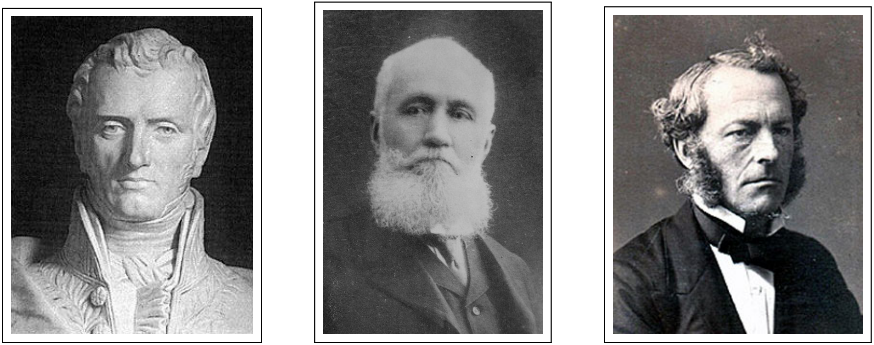
- Claude Navier (1822) and George Stokes (1845) figured out how to include the viscous force. The resulting equation for incompressible (constant ρ) hydrodynamics is called the Navier-Stokes (NS) equation.

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v}, \quad \text{with} \quad \nabla \cdot \mathbf{v} = 0.$$

- Here ν with dimensions of area per unit time is the coefficient of kinematic viscosity. NS needs to be supplemented with boundary conditions. At a solid boundary, the velocity must vanish, due to friction: this is the no-slip condition. Running a fan does not remove the dust accumulated on the blades.

- It is one of the important equations of physics, along with Newton's equations of celestial mechanics, Maxwell's equations of electromagnetism, Einstein's equations for gravity and Schrödinger's equation for an atom.

Claude Louis Navier, Saint Venant and George Stokes



Claude Louis Navier (left), Saint Venant (middle) and George Gabriel Stokes (right).

Motivating Navier Stokes: Heat diffusion equation

- Empirically it is found that the heat flux between bodies grows with the temperature difference. Fourier's law of heat diffusion states that the heat flux density vector (energy crossing unit area per unit time) is proportional to the negative gradient in temperature

$$\mathbf{q} = -k\nabla T \quad \text{where} \quad k = \text{thermal conductivity.} \tag{9}$$

- Consider gas in a fixed volume V. The increase in internal energy $U = \int_V \rho c_v T d\mathbf{r}$ must be due to the influx of heat across its surface S.

$$\int_V \partial_t(\rho c_v T) d\mathbf{r} = -\int_S \mathbf{q} \cdot \hat{n}\, dS = \int_S k\nabla T \cdot \hat{n}\, dS = k \int_V \nabla \cdot \nabla T\, d\mathbf{r}. \tag{10}$$

  $c_v =$ specific heat/mass (at constant volume, no work) and $\rho =$ density.

- V is arbitrary, so integrands must be equal. Heat equation follows:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad \text{where} \quad \alpha = \frac{k}{\rho c_v} \quad \text{is thermal diffusivity.} \tag{11}$$

- Heat diffusion is dissipative, temperature differences even out and heat flow stops at equilibrium temperature. It is not time-reversal invariant.

Including viscosity: Navier-Stokes equation

- Heat equation $\partial_t T = \alpha \nabla^2 T$ describes diffusion from hot $\to$ cold regions.
- (Shear) viscosity causes diffusion of velocity from a fast layer to a neighbouring slow layer of fluid. The viscous stress is $\propto$ velocity gradient. If a fluid is stirred and left, viscosity brings it to rest.
- By analogy with heat diffusion, velocity diffusion is described by $\nu \nabla^2 \mathbf{v}$.
- Kinematic viscosity ν has dimensions of diffusivity (areal velocity L^2/T).
- *Postulate* the Navier-Stokes equation for viscous *incompressible* flow:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \qquad (\mathrm{NS}). \tag{12}$$

- NS has not been derived from molecular dynamics except for dilute gases. It is the simplest equation consistent with physical requirements and symmetries. It's validity is restricted by experiment.
- NS is second order in space derivatives unlike the inviscid Euler eqn. Experimentally relevant boundary condition is impenetrability $\mathbf{v} \cdot \hat{n} = 0$ and 'no-slip' $\mathbf{v}_{||} = 0$ on fixed solid surfaces.

Navier-Stokes equation: challenges

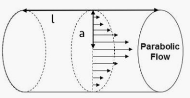
- Though simple to write down, the Navier-Stokes (NS) equation

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \qquad (\mathrm{NS}). \tag{13}$$

  is notoriously hard to solve in most physically interesting situations.
- A key issue is that the equation is non-linear in $\mathbf{v}$. Roughly, it is like the difference between trying to solve $2x + 3 = 0$ and $2x^7 + 3x^5 + 4x^4 + 9 = 0$.
- The conditions at boundaries and interfaces encode important physical effects, but can add to the complications. Ludwig Prandtl (1904) developed boundary layer theory for this.
- In fact, there is a million dollar Clay millenium prize attached to understanding some features of solutions to the NS equation.
- The challenge lies in deducing the observed, often complex, patterns of flow from the known laws governing fluid motion. This often requires a mix of physical insight, experimental data, mathematical techniques and computational methods.

Exact solutions: Creeping or Stokes flow

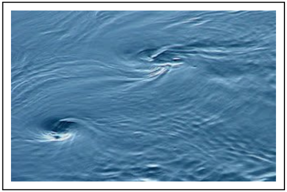
- Though the NS equation is very hard to solve in general, there are a few situations where exact solutions are available.
- This happens especially when the viscous force of dissipation is very large relative to inertial forces, as for instance in 'creeping flow' at very low flow speed. We recall two famous results.
- Poiseuille flow through a cylindrical pipe of length l and radius a due to a pressure drop Δp. The velocity profile is parabolic and the mass flowing through the pipe per unit time is $Q = \frac{\pi \Delta p}{8 \nu l} a^4$.



- Stokes studied steady constant density flow around a sphere of radius a moving at velocity $\mathbf{U}$ through a fluid with viscosity ν. He found the drag force on the sphere: $\mathbf{F}_{\rm drag} = -6\pi\rho\nu a\mathbf{U}$. Viscous drag is proportional to speed at low speeds. At higher speeds, there are deviations (the drag can be quadratic in velocity) as the flow ceases to be laminar.

Eddies and Vorticity

- *Vorticity* is a measure of local *rotation/angular momentum* in a flow. A flow without vorticity is called *irrotational*.
- Vortices are manifestations of vorticity in a flow.
- Vortices are ubiquitous in flows.
- We have many names for bananas: Vazhai, Kela, Puvan, Malapazham, Mondhan, Rasthali, Nendran, Yelakki, Karpuravalli, Chevvazhai, Musa, Virupakshi, Robusta, Udhayam etc.

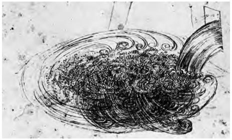


- Similarly, there are many names for vortex-like structures: swirls, eddies, vortices, whirlpools, whorls, cyclones, hurricanes, tornadoes, typhoons, maelstroms etc.
- Vortices can be created easily and put to good used, as this video by Walter Lewin indicates.

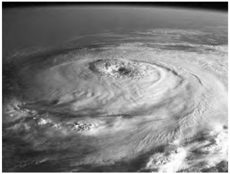
Leonardo da Vinci and vortices

- da Vinci was fascinated by vortices: many of his sketches contain detailed illustrations of eddies in fluids.





- Eddies can be of various sizes: in a sink, in the sea and in the atmosphere.

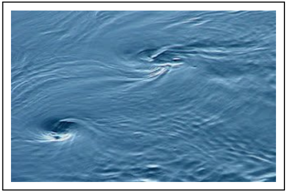



Leonardo da Vinci and vortices

- He even noticed similarities between vortices in the wake behind a flat plate and braided hair!

Vorticity and circulation

- *Vorticity* is a vector field, defined as $\mathbf{w} = \nabla \times \mathbf{v}$. It measures local *rotation/angular momentum* in a flow.
- Vorticity has dimensions of a frequency $[\mathbf{w}] = 1/T$.
- Given a closed contour C in a fluid, the circulation around the contour $\Gamma(C) = \oint_C \mathbf{v} \cdot d\mathbf{l}$ measures how much $\mathbf{v}$ 'goes round' C. By Stokes' theorem, it equals the flux of vorticity across a surface that spans C.

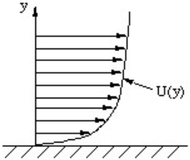


$$\Gamma(C) = \oint_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot dS = \int_S \mathbf{w} \cdot dS \quad \text{where} \quad \partial S = C.$$

- Enstrophy $\int \mathbf{w}^2 \, d\mathbf{r}$ measures global vorticity. It is conserved in ideal 2d flows, but not in 3d: it can grow due to 'vortex stretching' (see below).

Examples of flow with vorticity $\mathbf{w} = \nabla \times \mathbf{v}$

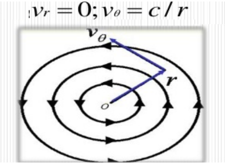
- Shear flow with horizontal streamlines is an example of flow with vorticity: $\mathbf{v}(x,y,z) = (U(y),0,0)$. Vorticity $\mathbf{w} = \nabla \times \mathbf{v} = -U'(y)\hat{z}$.



- A bucket of fluid rigidly rotating at small angular velocity $\Omega\hat{z}$ has $\mathbf{v}(r,\theta,z) = \Omega\hat{z} \times \mathbf{r} = \Omega r\hat{\theta}$. The corresponding vorticity $\mathbf{w} = \nabla \times \mathbf{v} = \frac{1}{r}\partial_r(rv_\theta)\hat{z}$ is constant over the bucket, $\mathbf{w} = 2\Omega\hat{z}$.

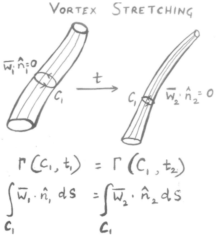
- The planar azimuthal velocity profile $\mathbf{v}(r,\theta) = \frac{c}{r}\hat{\theta}$ has circular streamlines. It has no vorticity $\mathbf{w} = \frac{1}{r}\partial_r(r\frac{c}{r})\hat{z} = 0$ except at $r = 0$: $\mathbf{w} = 2\pi c\delta^2(\mathbf{r})\hat{z}$. The constant $2\pi c$ comes from requiring the flux of $\mathbf{w}$ to equal the circulation of $\mathbf{v}$ around any contour enclosing the origin

$$\oint \mathbf{v} \cdot d\mathbf{l} = \oint (c/r) r\, d\theta = 2\pi c.$$



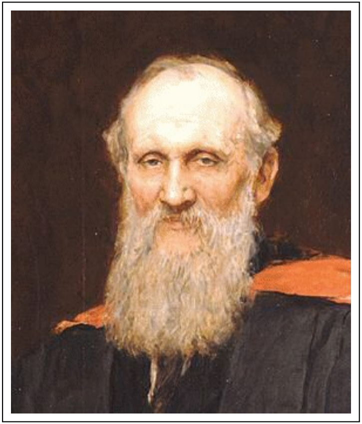
Vortex rings and tubes

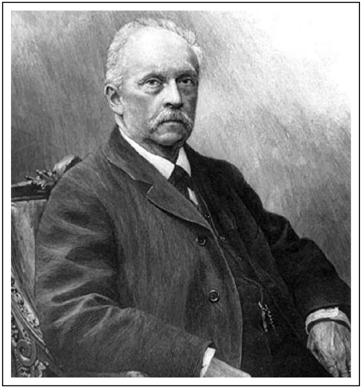
- Vortices can take the shape of tubes and rings. Kelvin and Helmholtz discovered many interesting properties of vortex tubes.
- Smoke rings are examples of vortex tubes. Dolphins blow vortex rings in water and chase them.



- Fluid flow tends to stretch and bend vortex tubes while carrying them along. They survive in the absence of viscosity but dissipate due to friction as seen in this video.

Lord Kelvin and Hermann von Helmholtz





Lord Kelvin (left) and Hermann von Helmholtz (right).

Evolution of vorticity and Kelvin's theorem

- Taking the curl of the Euler equation $\partial_t \mathbf{v} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \left(h + \frac{1}{2}\mathbf{v}^2\right)$ allows us to eliminate the pressure term in barotropic flow to get

$$\partial_t \mathbf{w} + \nabla \times (\mathbf{w} \times \mathbf{v}) = 0. \tag{14}$$

- This may be interpreted as saying that vorticity is 'frozen' into $\mathbf{v}$.
- The flux of $\mathbf{w}$ through a surface moving with the flow is constant in time:

$$\frac{d}{dt}\int_{S_t} \mathbf{w} \cdot dS = 0 \quad \text{or by Stokes' theorem} \quad \frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \frac{d\Gamma}{dt} = 0. \tag{15}$$

- Here C_t is a closed material contour moving with the flow and S_t is a surface moving with the flow that spans C_t.
- The proof uses the Leibnitz rule for material derivatives $D_t \equiv \partial_t + \mathbf{v} \cdot \nabla$

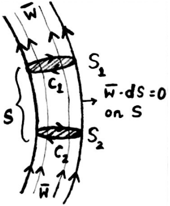
$$\frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \oint_{C_t} D_t \mathbf{v} \cdot d\mathbf{l} + \oint_{C_t} \mathbf{v} \cdot D_t d\mathbf{l}. \tag{16}$$

  Using the Euler equation $D_t \mathbf{v} = -\nabla h$ and $D_t d\mathbf{l} = d\mathbf{v}$ we get

$$\frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \oint_{C_t} d\left(\frac{1}{2}\mathbf{v}^2 - h\right) = 0. \tag{17}$$

Kelvin & Helmholtz theorems on vorticity

- $\frac{d}{dt}\oint_{C_t} \mathbf{v}\cdot d\mathbf{l} = 0$ is Kelvin's theorem: circulation around a *material* contour is constant in time. In particular, in the absence of viscosity, eddies and vortices cannot develop in an initially irrotational flow (i.e. $\mathbf{w} = 0$ at $t = 0$).
- Vortex tubes are cylindrical surfaces everywhere tangent to $\mathbf{w}$. So on a vortex tube, $\mathbf{w}\cdot dS = 0$.



- The circulation Γ around a vortex tube is independent of the choice of encircling contour. Consider part of a vortex tube S between two encircling contours C_1 and C_2 spanned by surfaces S_1 and S_2.
  - Applying Stokes' theorem to the closed surface $Q = S_1 \cup S \cup S_2$ we get

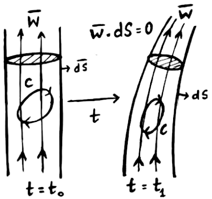
$$\int_Q \mathbf{w}\cdot dS = \int_{\partial Q} \mathbf{v}\cdot d\mathbf{l} = 0 \quad \text{as} \quad \partial Q \quad \text{is empty,}$$

$$\Rightarrow \int_{S_1} \mathbf{w}\cdot dS - \int_{S_2} \mathbf{w}\cdot dS = 0 \quad \text{or} \quad \Gamma(C_1) = \Gamma(C_2) \quad \text{since } \mathbf{w}\cdot dS = 0 \text{ on S}$$

  - As a result, a vortex tube cannot abruptly end, it must close on itself to form a ring (e.g. a smoke ring) or end on a boundary.

Helmholtz's theorem: inviscid flow preserves vortex tubes

- Suppose we have a vortex tube at initial time t_0. Let the material on the tube be carried by flow till time t_1. We must show that the new tube is a vortex tube, i.e., that vorticity is everywhere tangent to it, or $\mathbf{w} \cdot d\mathbf{S} = 0$.



- Consider a contractible closed curve $C(t_0)$ lying on the initial vortex tube, the flow maps it to a contractible closed curve $C(t_1)$ lying on the new tube. By Kelvin's theorem, $\Gamma(C(t_0)) = 0 = \Gamma(C(t_1))$. Now suppose S is the surface on the new vortex tube enclosed by $C(t_1)$, $\partial S = C(t_1)$, then

$$0 = \Gamma(C(t_1)) = \int_S \mathbf{w} \cdot d\mathbf{S}.$$

- This is true for any contractible closed curve $C(t_1)$ on the new tube. Considering an infinitesimal closed curve, we conclude that $\mathbf{w} \cdot d\mathbf{S} = 0$ at every point of the new tube, i.e., it must be a vortex tube.

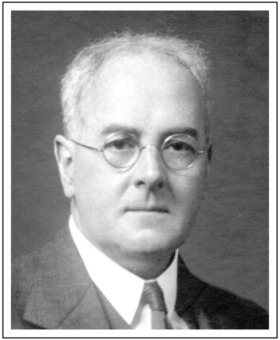
Vorticity: Creation, diffusion, stretching and cascade

- If there is no vorticity initially in a flow, then it cannot develop in the absence of viscosity.
- Viscous forces, especially in a layer near solid boundaries, can generate vorticity.
- Vorticity can diffuse through a flow and spread out.
- Vortex tubes tend to stretch and become narrower. As the flow develops, energy in larger vortices cascades to smaller ones. Vortices are finally destroyed by viscosity at the Taylor microscale.
- This was nicely captured in a poem by L F Richardson in *Weather Prediction by Numerical Process* (1922):

  Big whorls have little whorls that feed on their velocity,
  and little whorls have lesser whorls and so on to viscosity.

- Video of vortex ring collisions.

Lewis Fry Richardson

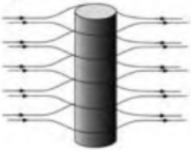


Big whorls have little whorls that feed on their velocity,
and little whorls have lesser whorls and so on to viscosity.

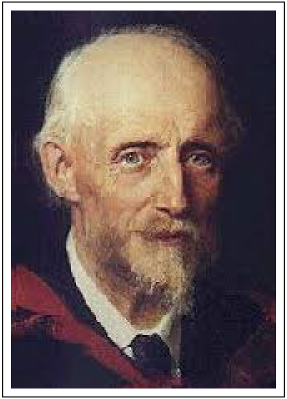
– L F Richardson, Weather Prediction by Numerical Process (1922).

Reynolds number $\mathscr{R}$ and similarity principle

- Suppose we consider water with uniform velocity $U\hat{x}$ flowing down a broad and deep channel. It meets a cylindrical obstacle of diameter L and flows round it creating a pattern.



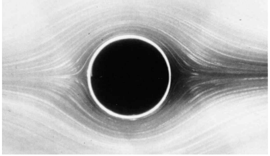
- It turns out that if we double the speed U and halve the radius a, then the same flow pattern results. This is the 'similarity' principle named after Osborne Reynolds who did careful experiments with fluids flowing down a pipe in the late 1800s.
- Incompressible flows with the same Reynold's number $\mathscr{R}$ look the same (the flows need not be laminar). $\mathscr{R} = LU/\nu$ is a dimensionless parameter that is a measure of the ratio of inertial to viscous forces.
- Flow around an aircraft is simulated in wind tunnels using a scaled down aircraft with the same $\mathscr{R}$.
- When $\mathscr{R}$ is small (e.g. in slow creeping flow), viscous forces dominate and the flow is regular. Interesting things happen as the flow speeds up and $\mathscr{R}$ increases!



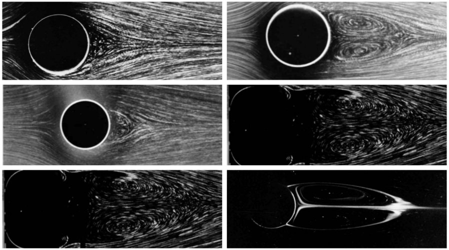
Osbourne Reynolds

Flow past a cylinder

- Consider flow with asymptotic velocity $U\hat{x}$ past a fixed cylinder of diameter L and axis along $\hat{z}$. The components of velocity are (u, v, w).
- At very low $\mathscr{R} \approx .16$, the symmetries of the (steady) flow are (a) $y \to -y$ (reflection in $z - x$ plane), (b) time and z translation-invariance (c) left-right symmetry w.r.t. center of cylinder ($x \to -x$ and $(u, v, w) \to (u, -v, -w)$).



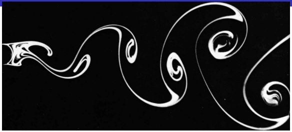
- All these are symmetries of Stokes flow (ignoring the non-linear advection term).
- At $\mathscr{R} \approx 1.5$ a marked left-right asymmetry develops.

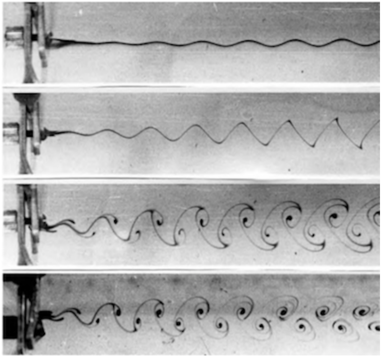


- At $\mathscr{R} \approx 4$, change in topology of flow: flow separates and recirculating standing eddies (from diffusion of vorticity) form downstream of cylinder.
- At $\mathscr{R} \approx 40$, flow ceases to be steady, but is periodic: undulating wake.

von Karman vortex street

- At $\mathscr{R} \gtrsim 50$, recirculating eddies are periodically (alternatively) shed to form the celebrated von Karman vortex street as shown in this video.





Theodore von Kármán

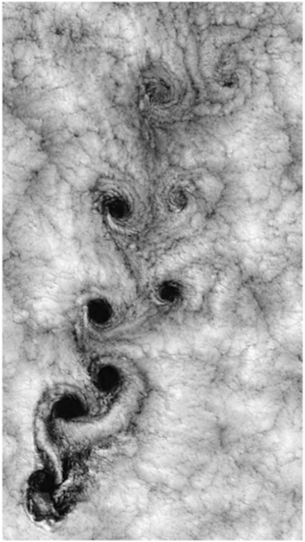
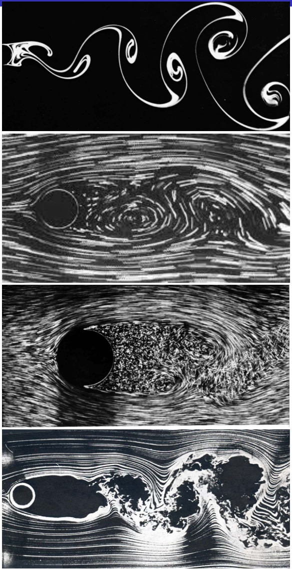


Fig. 2.8: A vortex street in clouds due to disruption of an atmospheric flow. (Photo: NOAA/ University of Maryland Baltimore County, Atmospheric Lidar Group.)

von Karman

Transition to turbulence in flow past a cylinder

- At $\mathscr{R} \gtrsim 40$, the vortex street develops with paired vortices being shed alternatively.
- The z-translation invariance is spontaneously broken when $\mathscr{R} \sim 40-75$.
- As $\mathscr{R}$ increases, some vortices lose their identity, vortex street is interspersed by turbulent patches.
- At $\mathscr{R} \sim 200$, flow becomes chaotic with turbulent boundary layer with vortex street persisting only close to the cylinder.
- At $\mathscr{R} \sim 1800$, only about two vortices in the von Karman vortex street are distinct before merging into a quasi uniform turbulent wake.
- At much higher $\mathscr{R}$, many of the symmetries of NS are restored in a statistical sense and turbulence is called fully-developed.



Practical consequences: Drag on a sphere in creeping flow

- Flow past a cylinder can be used to model drag force on car/plane/ship.
- Stokes studied incompressible (constant ρ) flow around a sphere of radius a moving through a viscous fluid with velocity $\mathbf{U}$

$$\mathbf{v}'_t + \mathbf{v}' \cdot \nabla' \mathbf{v}' = -\frac{1}{\rho} \nabla' p + \frac{1}{\mathscr{R}} \nabla'^2 \mathbf{v}, \quad \frac{1}{\mathscr{R}} = \frac{\nu}{aU} \tag{18}$$

- For steady flow $\partial_t \mathbf{v}' = 0$. For creeping flow ($\mathscr{R} \ll 1$) we may ignore advection term and take a curl to eliminate pressure to get

$$\nabla'^2 \mathbf{w}' = 0. \tag{19}$$

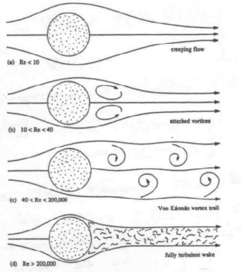
- By integrating the stress over the surface Stokes found the drag force

$$F_i = -\int \sigma_{ij} n_j dS \quad \Rightarrow \quad \mathbf{F}_{\text{drag}} = -6\pi\rho\nu a\mathbf{U}. \tag{20}$$

- Upto 6π factor, this follows from dimensional analysis! Magnitude of drag force is $F_D = \frac{12}{\mathscr{R}} \times \frac{1}{2}\pi a^2 \rho U^2$. For Stokes flow, drag coefficient is $12/\mathscr{R}$: this is experimentally verified.

Drag on a sphere at higher Reynolds number $\mathscr{R} = \frac{Ua}{\nu}$

- At higher speeds ($\mathscr{R} \gg 1$), naively expect viscous term to be negligible. However, experimental flow is far from ideal (inviscid) flow!
- At higher $\mathscr{R}$, flow becomes unsteady, vortices develop and a turbulent wake is generated.
- Dimensional analysis implies drag force on a sphere is expressible as $F_D = \frac{1}{2} C_D(\mathscr{R})\, \pi a^2 \rho U^2$, where $C_D = C_D(\mathscr{R})$ is the dimensionless drag coefficient, determined by NS equation.
- F can only depend on ρ, U, a, ν. To get mass dimension correctly, $F \propto \rho U^b \nu^c a^d$. Dimensional analysis $\Rightarrow b = d$ and $c = 2 - d$, so $F \propto \rho \left(\frac{Ua}{\nu}\right)^d \nu^2$. Thus, $F = C'_D(\mathscr{R})\, (\rho a^2 U^2)/\mathscr{R}^2$ $= \frac{1}{2} C_D(\mathscr{R})\, \pi a^2 \rho U^2$.
- Comparing with Stokes' formula for creeping flow $F = 6\pi a \rho \nu U$ we get $C_D \sim 12/\mathscr{R}$ as $\mathscr{R} \to 0$.



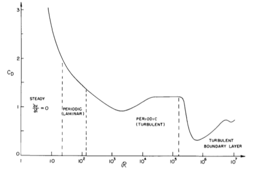
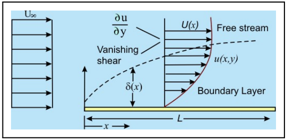


Fig. 41–4. The drag coefficient C_D of a circular cylinder as a function of the Reynolds number.

- Significant experimental deviations from Stokes' law: enhancement of C_D at higher $1 \leq \mathscr{R} \leq 10^5$, then drag force drops with increasing U!

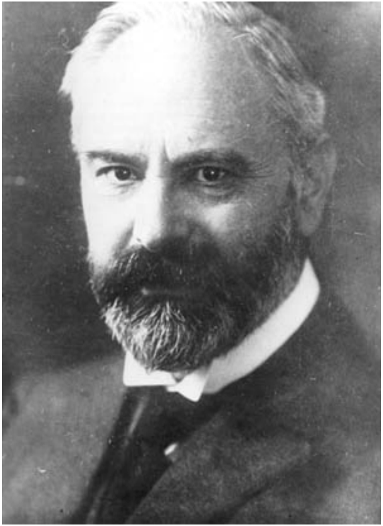
Drag crisis clarified by Prandtl's boundary layers

- In inviscid flow (Euler equation) tangential velocity on solid surfaces is unconstrained, can be large.
- For viscous Navier-Stokes flow, no slip boundary condition implies tangential $\mathbf{v} = 0$ on solid surfaces.
- Even for low viscosity, there is a thin boundary layer where tangential velocity drops rapidly to zero. In the boundary layer, cannot ignore $\nu \nabla^2 \mathbf{v}$.



- Though upstream flow is irrotational, vortices are generated in the boundary layer due to viscosity. These vortices are carried downstream in a (turbulent) wake.
- Larger vortices break into smaller ones and so on, due to inertial forces. Small vortices (at the Taylor microscale) dissipate energy due to viscosity increasing the drag for moderate $\mathscr{R}$.

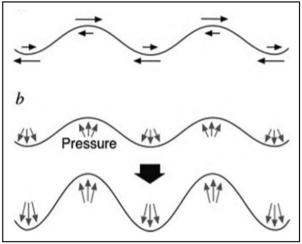
Ludwig Prandtl



Ludwig Prandtl (1875-1953)

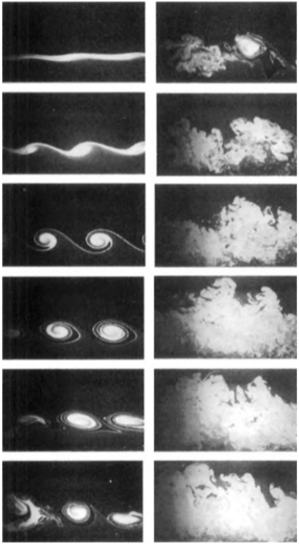
Kelvin Helmholtz instability

- Why does a regular laminar flow become turbulent when the Reynolds number is increased?
- The laminar flow pattern is unstable to perturbations. Instabilities lead to the growth of perturbations resulting in an alteration of the flow pattern.
- The Kelvin-Helmholtz shear flow instability is a prototype. It occurs when two neighbouring layers of fluid travel at different speeds. The flat interface becomes wavy, leading to the generation of eddies as seen in this video.



Development of KH instability (Flow, P. Ball)

Kelvin-Helmholtz instability: Roll-up of vortex sheet



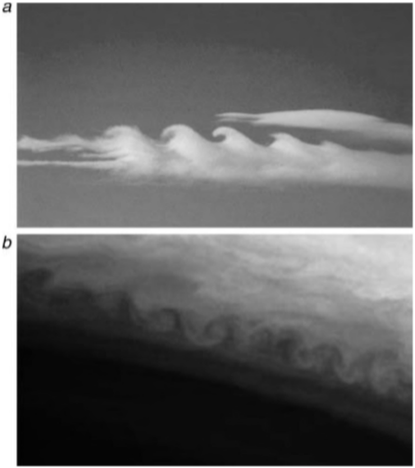
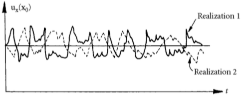


Fig. 2.11: A Kelvin–Helmholtz instability in atmospheric clouds (*a*), and in the atmosphere of Saturn (*b*). (Photos: *a*, Brooks Martner, NOAA/ Forecast Systems Laboratory; *b*, NASA.)

Left: KH instability development made visible by injecting dye into the interface and photographed by K R Sreenivasan.

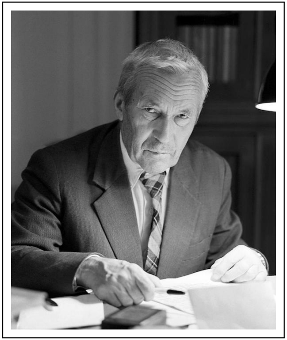
What is turbulence? Key features.

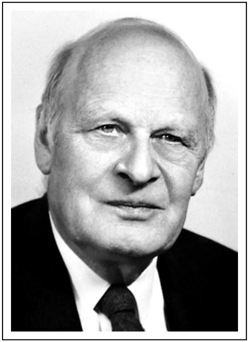
- Slow flow or very viscous fluid flow tends to be regular & smooth (laminar). If viscosity is low or speed sufficiently high ($\mathscr{R}$ large enough), irregular/chaotic motion sets in: streamlines get convoluted as in this video.
- Turbulence is chaos in a driven dissipative system with many degrees of freedom. Without a driving force (say stirring), the turbulence decays.
- $\mathbf{v}(\mathbf{r}_0, t)$ appears random in time and highly disordered in space.
- Turbulent flows exhibit a wide range of length scales: from the system size, size of obstacles, through large vortices down to the smallest ones at the Taylor microscale (where dissipation occurs).
- $\mathbf{v}(\mathbf{r}_0, t)$ are very different in distinct experiments with approximately the same ICs/BCs. But the time average $\bar{\mathbf{v}}(\mathbf{r}_0)$ is the same in all realizations.



- Unlike individual flow realizations, statistical properties of turbulent flow are reproducible and determined by ICs and BCs.
- As $\mathscr{R}$ is increased, symmetries (rotation/reflection/translation) are broken, but can be restored in a statistical sense in fully developed turbulence.

Andrei Kolmogorov and Lars Onsager

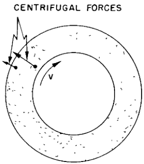




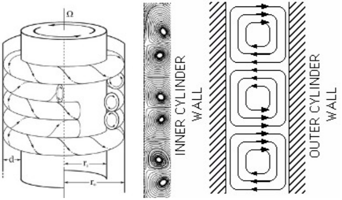
Andrei Kolmogorov (L) and Lars Onsager (R).

Taylor experiment: flow between rotating cylinders

- Oil with Al powder between concentric cylinders $a \leq r \leq b$. Inner cylinder rotates slowly at ω_a with outer cylinder fixed. Oil flows steadily with azimuthal v_ϕ dropping radially outward from $\omega_a r_a$ to zero at $r = b$.
- Shear viscosity transmits v_ϕ from inner cylinder to successive layers of fluid. Centrifugal force tends to push inner layers outwards, but inward pressure due to wall and outer layers balance it. So pure azimuthal flow is stable.

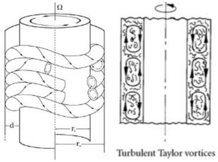


- When $\omega_a \approx \omega_{critical}$, flow is unstable to formation of toroidal Taylor vortices superimposed on the circumferetial flow. Translation invariance with z is lost. Fluid elements trace helical paths.
- Above $\omega_{critical}$, inward pressure and viscous forces can no longer keep centrifugal forces in check. The outer layer of oil prevents the whole inner layer from moving outward, so the flow breaks up into horizontal Taylor bands.

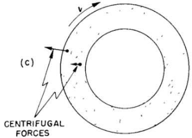


Taylor experiment: flow between rotating cylinders

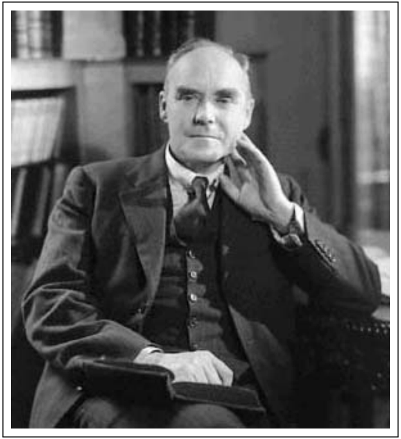
- If ω_a is further increased, keeping $\omega_b = 0$ then # of bands increases, they become wavy and go round at $\approx \omega_a/3$. Rotational symmetry is further broken though flow remains laminar.



- At sufficiently high ω_a, flow becomes fully turbulent but time average flow displays approximate Taylor vortices and cells.
- There are 3 convenient dimensionless combinations in this problem: $(b-a)/a$, L/a and the Taylor number $Ta = \omega_a^2 a(b-a)^3/\nu^2$.
- For small annular gap and tall cylinders $(L \gg a)$, Taylor number alone determines the onset of Taylor vortices at $Ta = 1.7 \times 10^4$.
- If the outer cylinder is rotated at ω_b holding inner cylinder fixed ($\omega_a = 0$), no Taylor vortices appear even for high ω_b. Pure azimuthal flow is stable.
- When outer layers rotate faster than inner ones, centrifugal forces build up a pressure gradient that maintains equilibrium.



Geoffrey Ingram Taylor



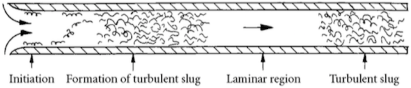
Geoffrey Ingram Taylor

Reynolds' expt (1883): Pipe flow transition to turbulence

- Consider flow in a pipe with a simple, straight inlet. Define the Reynolds number $\mathscr{R} = Ud/\nu$ where pipe diameter is d and U is flow speed.

- At very low $\mathscr{R}$ flow is laminar: steady Poiseuille flow (parabolic vel. profile).
- In general, turbulence in the pipe seems to originate in the boundary layer near the inlet or from imperfections in the inlet.

- If $\mathscr{R} \lesssim 2000$, any turbulent patches formed near the inlet decay.

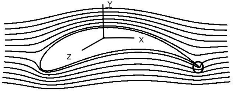


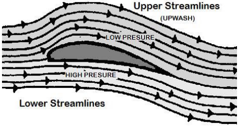
- When $\mathscr{R} \gtrsim 10^4$ turbulence first begins to appear in the annular boundary layer near the inlet. Small chaotic patches develop and merge until turbulent 'slugs' are interspersed with laminar flow regions.

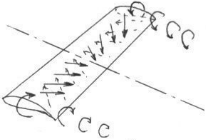
- For $2000 \lesssim \mathscr{R} \lesssim 10,000$, the boundary layer is stable to small perturbations. But finite amplitude perturbations in the boundary layer are unstable and tend to grow along the pipe to form fully turbulent flow.

Lift on an airfoil

- Consider an infinite airfoil of uniform cross section (axis along z). Airflow around it can be treated as 2-dimensional, i.e. on x, y plane.
- Airfoil starts from rest moves left with zero initial circulation. Ignoring $\nu\nabla^2\mathbf{v}$, Kelvin's theorem precludes any circulation developing around wing. Streamlines of potential flow have a singularity as shown in Fig 1.
- Viscosity at rearmost point due to large $\nabla^2\mathbf{v}$ regularizes flow pattern as shown in Fig.2.
- In fact, circulation Γ develops around airfoil (Fig. 3). In frame of wing, we have an infinite airfoil with circulation Γ placed perpendicularly in a rightward velocity field $v_\infty\hat{x}$.
- Situation is analogous to infinite wire carrying current I placed perpendicularly in a $\mathbf{B}$ field!





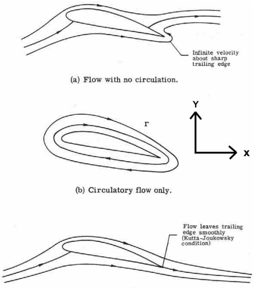


Circulation around and airfoil

- Current $\mathbf{j}$ in $\mathbf{B}$ field feels Lorentz force/Vol. $\mathbf{j} \times \mathbf{B}$ where $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$ by Ampere's law. Analogue of Lorentz force is vorticity force in Euler equation

$$\rho \partial_t \mathbf{v} + \rho \mathbf{w} \times \mathbf{v} = -\rho \nabla \sigma + \rho \nu \nabla^2 \mathbf{v}$$

- $\mathbf{B} \leftrightarrow \rho \mathbf{v}$, $\mathbf{j} \leftrightarrow \mathbf{w}$, $\mu_0 \leftrightarrow \rho$, $I \leftrightarrow \Gamma$. Current carrying wire feels transverse force BI/length. Expect airfoil to feel force $\rho v_\infty \Gamma$/length upwards ($\hat{y}$).



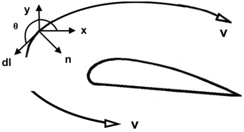
(a) Flow with no circulation.
(b) Circulatory flow only.
(c) Flow with circulation.

- Outside the boundary layer flow can be approximated as ideal irrotational flow which can be represented by a complex velocity $g = u - iv$. Since g is analytic outside the airfoil, we can expand it in a Laurent series, $g = v_\infty + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$.

- Circulation around a closed streamline enclosing airfoil just outside boundary layer is $\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \oint g dz = \oint (udx + vdy) + i(udy - vdx)$ since $(udy - vdx) = 0$ along a streamline. Thus by Cauchy's residue theorem, $\Gamma = 2\pi i a_{-1}$.

Kutta-Zhukowski lift formula for incompressible flow

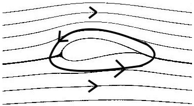
- Force exerted by flow on airfoil is $\mathbf{F} = \oint p\,\hat{n}\,dl$ where p is the air pressure along the boundary and $\hat{n}$ is the inward normal. By Bernoulli's theorem, $\oint p\,\hat{n}\,dl = -\frac{1}{2}\rho \oint \mathbf{v}^2\,\hat{n}\,dl$.



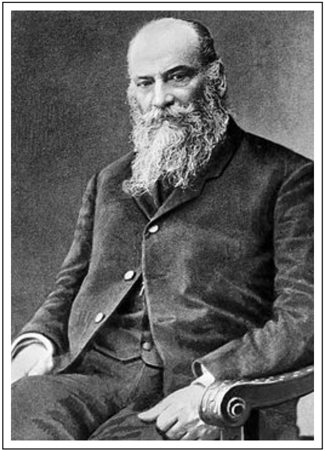
- If the line element $d\mathbf{l}$ along the streamline makes an angle θ with $\hat{x}$ then $(dx, dy) = (dl\cos\theta, dl\sin\theta)$ and the inward normal $\hat{n} = (-\sin\theta, \cos\theta)$. Thus, $F_x = \frac{1}{2}\rho \oint \mathbf{v}^2 \sin\theta\,dl = \frac{1}{2}\rho \oint \mathbf{v}^2\,dy$ and $F_y = -\frac{1}{2}\rho \oint \mathbf{v}^2 \cos\theta\,dl = -\frac{1}{2}\rho \oint \mathbf{v}^2\,dx$.
- The complex force $Z = F_y + iF_x = -\frac{\rho}{2}\oint \mathbf{v}^2(dx - idy)$ may be expressed in terms of the circulation Γ using the complex velocity g. As $udy - vdx = 0$,

$$\begin{aligned} Z &= -\frac{\rho}{2}\oint \left[\mathbf{v}^2(dx - idy) + 2i(udy - vdx)(u - iv)\right] = \frac{\rho}{2}\oint (v^2 - u^2 - 2iuv)(dx + idy) \\ Z &= -\frac{\rho}{2}\oint g^2 dz = -\frac{\rho}{2}\oint \left[v_\infty^2 + (2v_\infty a_{-1})/z + \cdots\right] dz = -(\rho/2)[2\pi i(2v_\infty a_{-1})] = -\rho v_\infty \Gamma \end{aligned}$$

  by Cauchy's theorem. So $F_x = 0$ and $F_y = -\rho v_\infty \Gamma$.
- $F_y > 0$ and generates lift if the counter-clockwise circulation Γ is negative, which is the case if speed above airfoil is more than below.



Nikolay Yegorovich Zhukovsky and Martin Wilhelm Kutta



Nikolay Yegorovich Zhukovsky (left) and Martin Wilhelm Kutta (right).

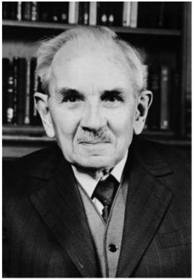
Shocks in compressible flow (Video of bullet shocks)

- A shock is usually a surface of small thickness across which $\mathbf{v}, p, \rho$ change significantly: shock front modelled as a surface of discontinuity.
- Shock moves faster than sound. Roughly, if shock propagates sub-sonically, it could emit sound waves ahead of the shock eliminating the discontinuity.
- Sudden localized explosions like supernovae or bombs often produce spherical shocks called blast waves. Nature of spherical blast wave from atom bomb was worked out by Sedov and Taylor in the 1940s.
- Suppose shock moves to the right in lab frame. In shock frame, shock front is at rest and material to the right rushes towards it at $v_1 > c_s$. Material from undisturbed pre-shock medium in front of shock (ρ_1) moves behind the shock to the post-shock medium to the left and gets compressed to $\rho_2 > \rho_1$.
- Fluxes of mass, momentum and energy are equal pre- and post-shock, relating ρ_1, v_1, p_1 to ρ_2, v_2, p_2 leading to Rankine-Hugoniot 'jump' conditions.
- Viscous term $\nu \nabla^2 \mathbf{v}$ is often important in the shock layer since $\mathbf{v}$ changes rapidly. Leads to heating of the gas in shock layer and entropy production.

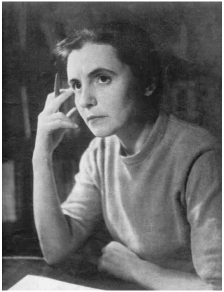
Existence & Regularity: Clay Millenium Problem

- Either prove the existence and regularity of solutions to incompressible NS subject to smooth initial data [in $\mathbb{R}^3$ or in a cube with periodic BCs] OR show that a smooth solution could cease to exist after a finite time.
- J Leray (1934) proved that weak solutions to Navier-Stokes exist, but need not be unique and could not rule out singularities.
- Hausdorff dim of set of space-time points where singularities can occur in NS cannot exceed one. So hypothetical singularities are rare!
- O Ladyzhenskaya (1969) showed existence and regularity of classical solutions to NS regularized with hyperviscosity $-\mu(-\nabla^2)^\alpha \mathbf{v}$ with $\alpha \geq 2$. J-L Lions (1969) extended it to $\alpha \geq 5/4$.
- A proof of existence/uniqueness/smoothness of solutions to NS or a demonstration of finite time blow-up is mathematically important.
- Physically, it is know that for large enough $\mathscr{R}$, most laminar flows are unstable, they become turbulent and seem irregular. Methods to calculate/predict features of turbulent flows would also be very valuable.

Jean Leray, Olga Ladyzhenskaya and Jacques Louis Lions



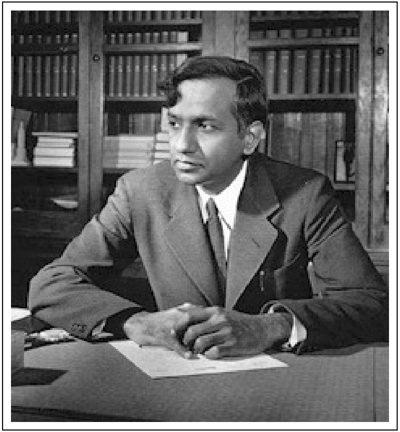
Jean Leray - Royal society (1991)

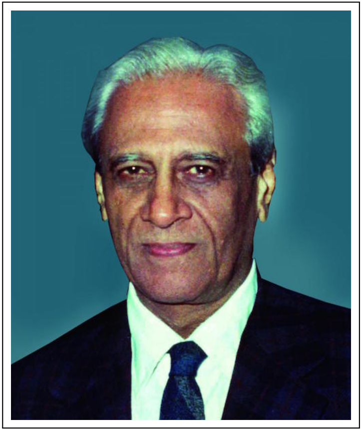


Jean Leray (left), Olga Ladyzhenskaya (middle) and Jacques Louis Lions (right).

Ladyzhenskaya Google doodle on her birth anniversary March 7, 2019.

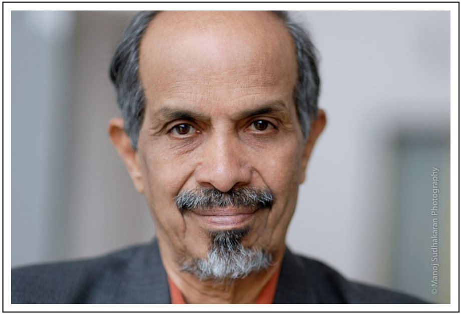
Prominent Indian fluid dynamicists

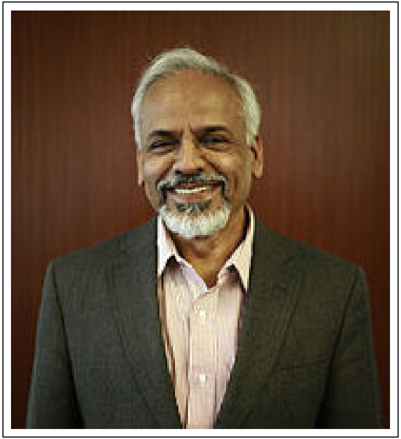




Subrahmanyan Chandrasekhar (left) and Satish Dhawan (right).

Prominent Indian fluid dynamicists





Vishnu Madav Ghatage, Roddam Narasimha (left) and Katepalli Sreenivasan (right).

von Karman vortex street in the clouds

von Karman vortex street in the clouds above Yakushima Island

Thank you!