Problems for Course on Integrable Systems
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1. For the harmonic oscillator show that $\{\theta, I\} = 1$ implies that $I'(H) = 1/\omega$. Here the angle variable is $\tan\theta = m\omega q/p$ and the proposed action I is a function of the Hamiltonian $H = (p^2/2m) + (m\omega^2 x^2/2)$. Proceed to find action-angle variables for the SHO and discuss the geometric meaning of the action.

2. Give a heuristic derivation of the bound state spectrum of the hydrogen atom using Bohr's quantisation condition for circular orbits using the conservation of energy $E = \mathbf{p}^2/2m - \alpha/r$, angular momentum and Laplace-Runge-Lenz vectors $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{A} = \mathbf{p} \times \mathbf{l} - m\alpha\hat{r}$. Hint: Show the relation $A^2 = 2mEl^2 + m^2\alpha^2$.

3. Show that the angular momentum Poisson algebra $\{L_i, L_j\} = \epsilon_{ijk} L_k$ is non-associative.

4. Find the frequency-wave vector dispersion relation $\omega = \omega(k)$ for (a) the wave equation $u_{tt} = c^2 u_{xx}$, (b) the linearized KdV equation $u_t - 6\bar{u}u_x + u_{xxx} = 0$ where $\bar{u}$ is a constant velocity, (c) the free particle 1d Schrödinger equation $i\hbar\psi_t = -(\hbar^2/2m)\psi''$ and (d) the 1d heat equation $u_t = \kappa u_{xx}$ where κ is the constant thermal diffusivity. Discuss the physical meaning of dispersion. Which equations describe dispersive and which non-dispersive propagation? How are the heat and Schrödinger equations related and how is it reflected in the dispersion relation and nature of time dependence of solutions?

5. **First alternate way of showing isospectrality of Lax matrix L:** Verify that the solution of the Lax equation $L_t = [L, A]$ with IC $L(0)$ may be expressed as $L(t) = SL(0)S^{-1}$ where $S(t)$ is a matrix satisfying the evolution equation $\dot{S} = -AS$ with IC $S(0) = I$. Use this solution to argue why the eigenvalues of L are time independent.

6. **Second alternate way of showing isospectrality of Lax matrix L:** Here we assume L is hermitian so that its eigenvalues λ are real: $L\psi = \lambda\psi$ and assume ψ is normalizable (with non-zero norm). Recall that we have shown that $(L - \lambda)(\psi_t + A\psi) = \lambda_t\psi$. Use hermiticity of L to show that $\lambda_t = 0$.

7. Show by explicit differentiation that $u(x,t) = -\frac{1}{2}c\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c}(x - ct - x_0)\right)$ for $c > 0$ is an exact solution of the KdV equation $u_t - 6uu_x + u_{xxx} = 0$. Plot the wave profile at $t = 0$ and justify the name solitary wave of depression. Discuss the direction of propagation. How are the wave depth, speed and width related? Hint: $(\mathrm{sech}\,x)' = -\mathrm{sech}\,x \tanh x$. How can you modify the KdV equation to get a very similar equation with a similar solution that is a solitary wave of elevation?

8. Show that the KdV equation $u_t - 6uu_x + u_{xxx} = 0$ is invariant under the scale transformation

$$x \to \mu x, \quad t \to \mu^3 t \quad \text{and} \quad u \to \mu^{-2} u \tag{1}$$

for any real constant $\mu \neq 0$.

9. Assuming decaying boundary conditions $|u| \to 0$ as $|x| \to \infty$, show that $\int_{-\infty}^{\infty} u\,dx$ is conserved under KdV evolution. What is the conserved current?

10. Use the KdV equation and integration by parts to show that $\int u^2 dx$ is a conserved quantity for the KdV equation with decaying boundary conditions. This quantity may be interpreted as momentum.

11. Show by differentiating in time and using the KdV equation, that $\int_{-\infty}^{\infty}(u^3 + \frac{1}{2}u_x^2)\, dx$ is conserved, again subject to decaying BCs. This quantity may be interpreted as energy.

12. **Lax Pair for 1st order linear wave equation.** We take $L = -\partial^2 + u$ to be the Schrödinger operator and A_1 to be a 1st order differential operator. L and A_1 are differential operators with coefficients that could depend on $u(x,t)$, possibly nonlinearly. They act on an auxiliary linear vector space of functions $\psi(x,t)$. When we say L_t we mean the derivative of L before it acts on ψ, so even if ψ depends on time, $L_t = u_t$. To be anti-symmetric, take $A_1 = c\partial + \text{h.c.}$ where $c(x,t)$ is an arbitrary function (to be fixed by requiring that the Lax equation is is consistent). (i) Show that $A_1 = [c, \partial]_+ = c_x + 2c\partial$. (ii) Show that

$$[L, A_1] = [-\partial^2 + u, c' + 2c\partial] = -c''' - 4c''\partial - 4c'\partial^2 - 2cu'. \tag{2}$$

(iii) Argue why c must be a constant and thereby show that the Lax equation $L_t = [L, A]$ is equivalent to the 1st order linear wave equation $u_t + 2cu_x = 0$ describing unidirectional propagation. (iv) For $c > 0$ does it describe right or left-moving waves?

13. **Lax pair for KdV:** Verify that $L = -\partial_x^2 + u(x,t)$ and $A = 4\partial_x^3 - 6u\partial_x - 3u_x + A_0(t)I$ is a Lax pair for the KdV equation for any function $A_0(t)$ of time alone. In other words, check that $L_t = [L, A]$ is equivalent to $u_t - 6uu_x + u_{xxx} = 0$.

14. **Riccati form of Schrodinger equation:** Show that the boosted Miura transformation between u and v: $u - \lambda = v_x + v^2$ (the Riccati equation) becomes a relation between u and ψ (the Schrödinger eigenvalue problem) $(-\partial^2 + u)\psi = \lambda\psi$ upon making the Cole-Hopf logarithmic derivative substitution $v = \psi_x/\psi$. This observation has nothing to do with KdV and predates KdV.

15. **mKdV to KdV:** Show that if you start with the KdV equation $u_t - 6uu_x + u_{xxx} = 0$ and make a Miura transform $u = v_x + v^2$, then v satisfies the 4th order equation

$$\left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2 v_x + v_{3x}) = 0. \tag{3}$$

Conclude that if v satisfies the mKdV (modified KdV) equation $v_t - 6v^2 v_x + v_{xxx} = 0$, then u satisfies the KdV equation.

16. **Lagrangian for KdV:** Show that the Euler-Lagrange equation for the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\phi_x\phi_t - \mathcal{H} = \frac{1}{2}\phi_x\phi_t - \phi_x^3 - \frac{1}{2}\phi_{xx}^2. \tag{4}$$

is the KdV equation $\phi_{xt} - 6\phi_x\phi_{xx} + \phi_{xxxx} = 0$ written in terms of the velocity potential ϕ where $\phi_x = u$. Hint: Recall that the EL equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_t \frac{\partial \mathcal{L}}{\partial \phi_t} + \partial_x \frac{\partial \mathcal{L}}{\partial \phi_x} - \partial_{xx} \frac{\partial \mathcal{L}}{\partial \phi_{xx}}. \tag{5}$$

17. **GGKM evolution equations for discrete spectrum of bound states:** The bound state wavefunctions $\phi_n(x)$ in the potential $u(x)$ (which goes to zero at $\pm\infty$) are eigenfunctions of $L = -\partial^2 + u$, $L\phi_n(x) = -\kappa_n^2 \phi_n(x)$ with eigenvalue $\lambda = -\kappa_n^2 < 0$. They are normalized to possess the asymptotic form

$$\phi_n(x) \to \begin{cases} e^{\kappa_n x} & \text{for } x \to -\infty \\ b_n e^{-\kappa_n x} & \text{for } x \to \infty. \end{cases} \tag{6}$$

Use the KdV Lax pair L, A with $A = 4\partial^3 - 3[u, \partial]_+$ to obtain the evolution equations

$$\dot{\kappa}_n(t) = 0 \quad \text{and} \quad \dot{b}_n(t) = 8\kappa_n^3 b_n(t) \tag{7}$$

Proceed by comparing the asymptotic behaviors as $x \to \pm\infty$ and using the fact that $\dot{\phi}_n + A\phi_n$ is also an eigenfunction of L with the same eigenvalue as ϕ_n.

18. **Direct scattering problem for attractive 1d Dirac delta potential**. Consider the 1d Schrodinger eigenvalue problem $(-\partial^2 + u)\phi = \lambda\phi$ for an attractive delta function potential $u(x) = -g\delta(x)$ with $g > 0$. (a) Get the continuity and discontinuity conditions on the eigenfunctions. (b) Show that there is one bound state with wavefunction

$$\phi_1(x) = e^{-g|x|/2} \quad \text{with} \quad \lambda_1 = -\kappa_1^2 \quad \text{where} \quad \kappa_1 = \frac{g}{2}. \tag{8}$$

Find the corresponding unit norm wave function $\tilde{\phi}_n$ and from its asymptotic behavior $\phi_n \to c_n e^{-\kappa_1 x}$ show that $c_1 = \sqrt{g}/2$.

(c) Consider scattering state wave functions with $\lambda = k^2$ written as

$$\phi_k(x) = \begin{cases} t(k)e^{-ikx} & \text{for} \quad x < 0 \\ e^{-ikx} + r(k)e^{ikx} & \text{for} \quad x > 0, \end{cases} \tag{9}$$

Find the reflection and transmission amplitudes. Argue that $1 + r = t$ and that

$$r(k) = -\frac{g}{g + 2ik} \quad \text{and} \quad t(k) = \frac{2ik}{g + 2ik}, \quad \text{so that} \quad a = \frac{1}{t} = 1 + \frac{g}{2ik} \quad \text{and} \quad b = \frac{r}{t} = \frac{ig}{2k}. \tag{10}$$

(d) Discuss the analytic structure of r, t, a, b on the complex k plane and relate it to the bound state spectrum.

19. **Inverse scattering for 1d Dirac delta via GLM equation.** Recall that the potential $u(x)$ may be determined from the scattering data by solving the Gelfand-Levitan-Marchenko equation

$$F(x) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \int_{-\infty}^{\infty} r(k) e^{ikx} \frac{dk}{2\pi}. \tag{11}$$

with $u(x) = -2\frac{d}{dx}K(x,x)$. We will illustrate the use of the GLM equation to recover the delta function potential from its scattering and bound state data. (a) Use complex contour integration to show that

$$F(x) = \frac{g}{2}e^{-\frac{gx}{2}} - \frac{g}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{g + 2ik} dk = \frac{g}{2}e^{-\frac{gx}{2}} - \frac{g}{2\pi}\pi e^{-\frac{gx}{2}}\Theta(x > 0) = \frac{g}{2}e^{-\frac{gx}{2}}\Theta(x < 0). \tag{12}$$

(b) Show that the GLM equation for the kernel K becomes

$$K(x,z)+\frac{g}{2}e^{-\frac{g}{2}(x+z)}\Theta(x+z<0)+\int_x^\infty K(x,y)\,\frac{g}{2}e^{\frac{g}{2}(y+z)}\Theta(y+z<0)dy=0. \tag{13}$$

(c) For $x+z>0$ show that both step functions vanish and that $K(x,z)=0$. (d) For $x+z<0$ show that the GLM equation becomes

$$K(x,z)+\frac{g}{2}e^{-\frac{g}{2}(x+z)}+\int_x^{-z} K(x,y)\frac{g}{2}e^{-\frac{g}{2}(y+z)}dy=0. \tag{14}$$

(e) Show that the solution is given by $K(x,z)=-g/2$ for $x+z<0$ so that $K(x,z)=-(g/2)\Theta(x+z<0)$ (there are general theorems that guarantee that the solution is unique). (f) Find the potential $u(x)$, and compare with the attractive delta potential we began with.

20. **Phase shift in scattering of KdV solitary waves:** Staring with the initial profile $u(x,0)=-6\,\mathrm{sech}^2\,x$, it is possible to use the IST to determine $u(x,t)$ at all times $-\infty<t<\infty$:

$$u(x,t)=-12\frac{[3+4\cosh(2x-8t)+\cosh(4x-64t)]}{\{3\cosh(x-28t)+\cosh(3x-36t)\}^2}. \tag{15}$$

At early and late times, one can show that $u(x,t)$ has the asymptotic forms

$$u(x,t)\sim-8\mathrm{sech}^2(2\xi\mp\frac{1}{2}\log 3)-2\mathrm{sech}^2(\eta\pm\frac{1}{2}\log 3)\quad\text{as}\quad t\to\pm\infty. \tag{16}$$

Here $\xi=x-16t$ and $\eta=x-4t$. Discuss the asymptotic nature of this solution of KdV with appropriate figures: how many solitons scatter, what are their speeds and shapes (tall/wide etc) and what are the phase shifts in the scattering.

21. **Recursion relation for KdV conserved densities.** Use the Gardner transform $\rho_x+\rho^2-2ik\rho=u(x,t)$ and Laurent-like asymptotic expansion $\rho=\sum_1^\infty\rho_n/(2ik)^n$ to derive the recursion relation

$$\rho_{n+1}=\partial_x\rho_n+\sum_{p=1}^{n-1}\rho_p\rho_{n-p} \tag{17}$$

and the 'initial' condition $\rho_1=-u$.

22. **First few conserved densities ρ_n for KdV:** Use the above recursion relation to show that

$$\begin{aligned}\rho_1=u,\quad \rho_2=-u_x,\quad \rho_3=u^2-u_{xx},\quad \rho_4=(2(u^2)-u_{xx})_x,\\ \rho_5=-u_{4x}+2(u^2)_{xx}+u_x^2+2uu_{xx}-2u^3.\end{aligned} \tag{18}$$

Show that $\int_{-\infty}^\infty\rho_3\,dx$ and $\int_{-\infty}^\infty\rho_5\,dx$ lead to the conserved momentum and energy of KdV.

23. **Show that ρ_{2n} are exact differentials ('total derivatives')**. Proceed by splitting $\rho=\rho_R+i\rho_I$. (a) Use the Gardner transform $\rho_x+\rho^2-2ik\rho=u$ to derive the equation $\rho_R(2\rho_I-2k)=-\partial_x\rho_I$. (b) Show that ρ_R is an exact differential. (c) Use the Laurent expansion $\rho=\sum_1^\infty\rho_n/(2ik)^n$ and the reality of ρ_n (see the next problem) to show that ρ_{2n} are exact differentials. Thus $\int\rho_{2n}\,dx=0$ do not give non-trivial conserved quantities.

24. **Reality of conserved densities ρ_n for KdV:** Recall that the conserved density $\rho(x,k)$ is related to the real KdV field u via the Gardner transform

$$\rho_x + \rho^2 - 2ik\rho = u(x,t) \tag{19}$$

which is a 1-parameter family of Riccati-type equations (labelled by k). We expanded ρ in a JWKB-like Laurent series

$$\rho = \sum_{n=1}^{\infty} \frac{\rho_n}{(2ik)^n}. \tag{20}$$

Argue now that ρ_n are real. Hint: Compare the equations satisfied by $\rho(x,k)$ and $\rho^*(x,-k)$, argue that they must both vanish as $x \to \infty$ and use uniqueness of solutions of the Gardner transform equation.

25. **KdV conserved charges H_n are in involution**. We claim that $H_n = ((-1)^{n+1}/2) \int_{-\infty}^{\infty} \rho_{2n+1} dx$ Poisson commute (with respect to both Poisson brackets). Here we will show it for the Gardner bracket, the same argument works for the Magri bracket. This would in essence make KdV an infinite dimensional Liouville integrable system. For this, we will use the bi-Hamiltonian structure of the KdV hierarchy:

$$\partial_{t_n} u(x) = \{u, H_{n+1}\}_1 = \{u, H_n\}_2 \tag{21}$$

where

$$\{u(x), u(y)\}_1 = \partial_x \delta(x-y) \quad \text{and} \quad \{u(x), u(y)\}_2 = \left(-\partial_x^3 + 2[\partial_x, u]_+\right) \delta(x-y) \tag{22}$$

are the Gardner and Magri brackets. (a) Use the Leibnitz rule to show that the Gardner bracket of any two functionals of u can be written as

$$\{F[u], G[u]\}_1 = \int dxdy \frac{\delta F}{\delta u(x)} \frac{\delta G}{\delta u(y)} \{u(x), u(y)\}_1 = -\int \left(\partial_x \frac{\delta F}{\delta u(x)}\right) \frac{\delta G}{\delta u(x)}\, dx = \int \frac{\delta F}{\delta u(x)} \partial_x \frac{\delta G}{\delta u(x)}\, dx. \tag{23}$$

Similarly, integrating by parts one has for the Magri bracket

$$\{F[u], G[u]\}_2 = \int \frac{\delta F}{\delta u(x)} \left(-\partial_x^3 + 2[\partial_x + u]_+\right) \frac{\delta G}{\delta u(x)}\, dx \tag{24}$$

(b) Thus, show that (21) becomes

$$\partial_x \frac{\delta H_{n+1}}{\delta u(x)} = \left(-\partial_x^3 + 2[\partial_x, u]_+\right) \frac{\delta H_n}{\delta u(x)}. \tag{25}$$

(c) Use this to show the 'step-down-step-up' formula for the conserved charges with respect to the Gardner bracket

$$\{H_p, H_q\}_1 = \{H_{p-1}, H_{q+1}\}_1. \tag{26}$$

(d) By repeated use of the step-down-step-up formula, show that $\{H_p, H_q\} = 0$. Hints: (i) First try a special case like $p = 1, q = 2$. (ii) Then take (without loss of generality) $p > q$ and suppose $p - q$ is even. (iii) then consider the case $p - q$ odd.

26. **Zero curvature representation for NLSE:** Show that the nonlinear Schrödinger equation for the complex scalar field ψ, $i\psi_t = -\psi_{xx} + 2\kappa|\psi|^2\psi$, which describes a 1d gas of bosons in a mean field approximation, admits a zero curvature representation $U_t - V_x + [U, V] = 0$ if U and V are chosen as the 2×2 matrices

$$U = \sqrt{\kappa}(\psi^* \sigma_+ + \psi \sigma_-) + \lambda \frac{\sigma_3}{2i} \qquad \text{and}$$

$$V = i\kappa|\psi|^2\sigma_3 - i\sqrt{\kappa}\left(\psi_x^*\sigma_+ - \psi_x\sigma_-\right) - \lambda\sqrt{\kappa}(\psi^*\sigma_+ + \psi\sigma_-) - \lambda^2\frac{\sigma_3}{2i}. \quad (27)$$

Here $\sigma_\pm = (1/2)(\sigma_1 \pm i\sigma_2)$ are built from the Pauli matrices σ_1 and σ_2. It is often convenient to write U and V as

$$\begin{aligned} U &= U_0 + \lambda U_1 \quad \text{and} \quad V = V_0 + \lambda V_1 + \lambda^2 V_2 \quad \text{where} \\ U_0 &= \sqrt{\kappa}(\psi^*\sigma_+ + \psi\sigma_-) = -V_1, \quad U_1 = \frac{\sigma_3}{2i} = -V_2, \\ V_0 &= i\kappa|\psi|^2\sigma_3 - i\sqrt{\kappa}\left(\psi_x^*\sigma_+ - \psi_x\sigma_-\right). \end{aligned} \quad (28)$$

and organise the calculation in powers of λ. Write out U and V as 2×2 matrices and proceed. Recall that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (29)$$

27. Check that NLSE follows from the **Hamiltonian and canonical Poisson brackets**

$$H = \int \left(|\psi_x|^2 + \kappa|\psi|^4\right) dx \quad \text{and} \quad \{\psi(x), \psi^*(y)\} = -i\delta(x-y). \quad (30)$$

28. **Path ordered exponential.** Consider the linear system of ODEs for the transition matrix $T(y,x)$:

$$\partial_y T(y,x) = U(y)T(y,x) \quad \text{with} \quad T(x,x) = I. \quad (31)$$

For each x and y, T is a matrix and so is $U(y)$. The difficulty in solving this equation lies in the fact that U is not a constant matrix and worse, the matrices $U(y)$ may not commute at distinct values of y. If they did, then the solution is just an ordinary exponential. (a) Convert this system of ODEs into an integral equation (what do you integrate with respect to and from where to where?). Show that you get

$$T(y,x) = I + \int_x^y U(z)T(z,x)\, dz. \quad (32)$$

(b) By repeated use ('Picard iteration') of this formula obtain a series representaion for the transition matrix

$$T(y,x) = \sum_{n=0}^{\infty} \int \cdots \int_{x<z_n<\cdots<z_1<y} dz_1 \cdots dz_n\, U(z_1)U(z_2)\cdots U(z_n). \quad (33)$$

(c) Now, define path ordering denoted by the symbol P via

$$\mathrm{P}(U(z_1)U(z_2)) = \begin{cases} U(z_1)U(z_2) & \text{if} \quad z_1 \geq z_2 \\ U(z_2)U(z_1) & \text{if} \quad z_2 \geq z_1, \end{cases} \quad (34)$$

Essentially the earlier locations are placed to the right. Argue that $\int_{z_1>z_2} dz_1 dz_2 U(z_1)U(z_2) = \int_{z_2>z_1} dz_1 dz_2 U(z_2)U(z_1)$, and thereby show that

$$\int_x^y dz_1 \int_x^{z_1} dz_2 U(z_1)U(z_2) = \frac{1}{2}\int_x^y dz_1 \int_x^y dz_2\, \mathrm{P}(U(z_1)U(z_2)). \quad (35)$$

Essentially, the integral over a triangle has been expressed as half the integral over a square. Proceeding this way, one can write the above series in a manner reminiscent of the exponential series

$$\int_x^y dz_1 \int_x^{z_1} dz_2 \cdots \int_x^{z_{n-1}} dz_n\, U(z_1)U(z_2)\cdots U(z_n) = \frac{1}{n!}\int_x^y \cdots \int_x^y dz_1 dz_2 \cdots dz_n \mathrm{P}(U(z_1)U(z_2)\cdots U(z_n)) \quad (36)$$

so that

$$T(y,x) = \sum_0^\infty \frac{1}{n!}\int_x^y \cdots \int_x^y dz_1 dz_2 \cdots dz_n \mathrm{P}(U(z_1)U(z_2)\cdots U(z_n)) =: \mathrm{P}\exp\left[\int_x^y U(z)dz\right]. \tag{37}$$

This series is called the path-ordered exponential and denoted $\mathrm{P}\exp$. The last expression is just a short form and is defined by the series.

29. Recall the **permutation operator** acting on a 3-fold tensor product: $P_{12}(u\otimes v\otimes w) = v\otimes u\otimes w$ etc. There are essentially 3 permutation operators P_{12}, P_{23}, P_{31} on $V\otimes V\otimes V$. (a) What about P_{21}, P_{13}, P_{32}? (b) Show that any one can be written in terms of the other two in two different ways: E.g.
$$P_{13} = P_{23}P_{12}P_{23} = P_{12}P_{23}P_{12}. \tag{38}$$

30. Here we will verify that the **NLSE classical r-matrix satisfies the classical Yang-Baxter equation**. We define
$$r_{ij}(\lambda_i - \lambda_j) = \frac{\kappa P_{ij}}{2(\lambda_i - \lambda_j)} \tag{39}$$
Introducing $r_{ij}\equiv r_{ij}(\lambda_i-\lambda_j)$ with $\lambda_1 = \lambda, \lambda_2 = \lambda'$ and $\lambda_3 = \lambda''$, the CYBE takes the compact form
$$\mathrm{CYBE} = [r_{12}, r_{23}] + [r_{23}, r_{31}] + [r_{31}, r_{12}] = 0. \tag{40}$$
Use the properties of the permutation operators to check that the CYBE is indeed satisfied.

31. **Bäcklund transformation for sine-Gordon and its kink soliton:** (a) In d'Alembert's ('light cone') coordinates $t, x = x\pm t$, show that the wave operator $\partial_t^2 - \partial_x^2 \propto \partial_t\partial_x$ upto a numerical factor. (b) In light-cone coordinates, the relativistic sine-Gordon field equation in appropriate units is $u_{xt} = \sin u$. Consider the Bäcklund relations
$$\frac{1}{2}(u+v)_x = a\sin((u-v)/2) \quad \text{and} \quad \frac{1}{2}(u-v)_t = \frac{1}{a}\sin((u+v)/2). \tag{41}$$
for a non-zero constant a. Use these relations to show that
$$\frac{1}{2}(u+v)_{xt} = \cos\frac{u-v}{2}\sin\frac{u+v}{2} \quad \text{and} \quad \frac{1}{2}(u-v)_{tx} = \cos\frac{u+v}{2}\sin\frac{u-v}{2}. \tag{42}$$
(c) Use $\sin(a+b) = \sin a\cos b + \cos a\sin b$ to show that u and v must satisfy the SG equation. Thus (41) may be regarded as an auto-Bäcklund transformation for SG. (d) Exploit this transformation to generate a non-trivial solution to SG starting from the trivial solution $v\equiv 0$. Itegrate the Backlund relations in this case to get
$$2ax = 2\log|\tan\frac{u}{4}| + f(t) \quad \text{and} \quad \frac{2t}{a} = 2\log|\tan\frac{u}{4}| + g(x) \tag{43}$$
where f and g are arbitrary functions of integration. This leads to the solution
$$u = 4\arctan\left(\kappa e^{ax+t/a}\right) \tag{44}$$
where κ is a constant of integration. This solution is called the sine-Gordon kink for $a>0$ and the anti-kink for $a<0$. Plot it to find out why. (e) Discuss in the context of the vacua of the SG field (what is the Lagrangian for the SG field equations).

32. **Painlevé II from similarity reduction of KdV:** Based on the previously discussed scaling symmetry of KdV, we observe that the combinations $t^{2/3}u$ and $xt^{-1/3}$ are scale-invariant. This suggests that we look for scaling solutions of KdV where $u(x,t) = -(3t)^{-2/3}f(\eta)$ where the scaling variable $\eta = x(3t)^{-1/3}$. Show that this leads to the 3rd order ODE $f''' + (6f - \eta)f' - 2f = 0$. Show that this ODE reduces to Painleve II ($w'' = 2w^3 + zw + \alpha$) by making the Miura transform-like substitution $f = \lambda w' - w^2$.

33. **Zero curvature representation for ASDYM:** If we introduce the coordinates $z, \tilde{z}, w, \tilde{w}$ and metric $ds^2 = 2(d\tilde{z}dz - d\tilde{w}dw)$ on 4d (complexified) space, then the anti-self-duality condition becomes
$$F_{wz} = 0, \quad F_{w\tilde{w}} = F_{z\tilde{z}} \quad \text{and} \quad F_{\tilde{w}\tilde{z}} = 0. \tag{45}$$
Here $F_{zw} = [D_z, D_w] = \partial_z A_w - \partial_w A_z + [A_z, A_w]$ etc. Find a zero curvature representation for these by proposing a pair of covariant derivatives L and M such that the condition for the corresponding curvature to vanish $R = [L, M] = 0$ for all values of the spectral parameter λ is equivalent to the ASDYM equations. Proceed by taking L and M to be suitable linear combinations of the covariant derivatives D_w, D_z, $D_{\tilde{z}}, D_{\tilde{w}}$ with suitable powers of λ.s