Introduction to Integrable Systems

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Contents

1 Some reference books/review articles

- P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, Cambridge University press, Cambridge (1989).
- A. Das, *Integrable models*, World Scientific Publishing Co. Pte. Ltd., Singapore (1989).
- L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Berlin (1987).
- S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of Solitons - The Inverse Scattering Method*, Consultants bureau, Plenum Publishing Corporation (1984).
- M. Dunajski, *Solitons, Instantons and Twistors*, Oxford University Press, Oxford (2010).
- G. Arutyunov, *Elements of Classical and Quantum Integrable Systems*, Springer, Cham (2019).
- S. Ruijsenaars, *Integrable Systems: An Overview*, https://www1.maths.leeds.ac.uk/~siru/ISsurvey.pdf.

Some more references and reviews (say, with a focus on quantum field theory, statistical mechanics, quantum S-matrices, spin chains, AdS/CFT etc) are listed in the Bibliography at the end of these notes.

2 Introduction

• What do we mean by an integrable or solvable model? Perhaps it is good to recall what we mean by a solvable algebraic equation. A polynomial equation of the $n^{\rm th}$ degree is called solvable if the roots can be expressed in terms of the coefficients using sums, differences, products, quotients and radicals (square-roots, cube-roots etc.). Classical results (especially of the 16^{th} century Italian school: Cardano, Tartaglia, Ferrari) provide such general formulae for the roots of polynomial equations up to the fourth degree. No such general formula is available for degree five and higher, though specific higher order polynomial equations (e.g. $z^5 - 1 = 0$) can be solved in closed form. Galois theory shows that such a general formula is impossible by relating the question to properties of groups of symmetries[1] of the equations (essentially the permutation groups S_n) that act by permuting the roots and to the nature of field extensions. This approach also explained why it is not possible to solve certain geometric problems such as trisecting an angle or 'doubling a cube' (constructing the edge of a cube with double the volume of a given cube) by ruler and compass. It is also noteworthy that the solvability of such problems depends on the tools that one is allowed to use. For instance, general formulae for the roots of a quintic can be found if one is allowed to use elliptic functions.

There have been attempts to develop a 'differential Galois theory' for ODEs though the idea has not been as fruitful. Sophus Lie, in fact, originally introduced Lie groups to discuss symmetries of ODEs. On the other hand, physically inspired ideas have led to several perhaps unexpected notions of solvability. In classical mechanics or classical field theory, solvability usually means we have some way of finding exact solutions to the equations of motion, which are typically ODEs or PDEs. In favorable cases, it means we can write down explicit solutions in terms of known functions (e.g. polynomial, exponential, trigonometric, hypergeometric, elliptic, Painlevé and other such special functions). However, even for models that we would like to designate as solvable, this may not be possible for arbitrary initial or boundary conditions. Sometimes, integrable means a reduction in complexity from PDEs to ODEs or from ODEs to quadrature (evaluating integrals) or algebraic equations or from nonlinear to linear systems etc. We will meet such examples. This reduction in complexity is often attributable to the presence of (sufficiently many) conserved quantities. Integrability also extends to quantum systems. Here, it could mean the possibility to exactly (or with reduced complexity) determine the energy levels and energy eigenfunctions as well as the scattering matrix. A nice feature of integrable systems is the deep interconnections that it reveals between the classical and quantum theories.

2.1 Integrability in Hamiltonian mechanics

A common theme in the theory of integrable systems is that of conserved quantities. Recall that for a particle moving on a line, Newton's generally nonlinear 2nd order ordinary differential equation $m\ddot{q} =$

[1] An algebraic equation is solvable by radicals if the corresponding Galois group is solvable. Roughly, a solvable group is one that can be built up from abelian groups by extensions. More precisely, the 'derived sequence' consisting of commutator subgroups $G^{(0)} = G, G^{(1)} = [G, G], G^{(2)} = [G^{(1)}, G^{(1)}], \cdots$ terminates in the trivial group after a finite number of steps. The successive quotients $G^{(j+1)}/G^{(j)}$ are abelian in this case and G may be viewed as being constructed by a succession of extensions of the trivial group by these abelian groups.

$-V'(q)$ can be reduced to quadrature using the conservation of energy[2] $E = \frac{1}{2}m\dot{q}^2 + V(q)$ allowing us to write $t - t_0 = \int_{q_0}^{q} \frac{dq'}{\sqrt{(2/m)(E-V(q'))}}$. While one still needs to do this integral and invert the result[3] to find q as a function of t, even such a reduction to quadrature is in general not possible for nonlinear systems with $n > 1$ degrees of freedom. For instance, if $m\ddot{q}^i = -\partial_i V(q)$ for $i = 1, 2, \cdots, n$, taking a dot product with the integrating factor $\dot{q}^i$ we get the conservation of $E = \frac{1}{2}m\sum_i \dot{q}^i\dot{q}^i + V(q)$. In this case though, we cannot in general separate variables as we did for $n = 1$. In fact for suitable potentials these problems include the great problems of celestial mechanics such as the 3 and n body problems, which are not known to be integrable: the solution of the EOM cannot be reduced to quadratures. Conserved quantities, however, can help. For a $2n$-dimensional phase space, each conserved quantity F forces trajectories to lie on a $2n-1$ dimensional hypersurface on which the conserved quantity takes a particular value (a level set of F). Each additional independent conserved quantity therefore constrains trajectories to a submanifold of phase space of one lower dimension. Evidently, the maximum possible number of independent conserved quantities that would allow for continuous time evolution is $2n - 1$. This, however, is very rare, even among systems we will call integrable. For instance, let us consider an Eulerian rigid body free to rotate about a fixed point. Euler's equations $\dot{\mathbf{L}} = \Omega \times \mathbf{L}$ describe motion in the 3d phase space with coordinates given by the cartesian components of angular momentum $L_{1,2,3}$ in the corotating frame of the body. Here $\mathbf{L} = I\Omega$ where I is the inertia tensor. This quadratically nonlinear system of 3 ODEs admits 2 independent conserved quantities the square of angular momentum $L^2 = \sum_i L_i L_i$ and the energy $H = \frac{1}{2}\mathbf{L}^t I^{-1}\mathbf{L}$. In the principal axis basis where I is diagonal with eigenvalues given by the principal moments of inertia I_1, I_2, I_3, $H = \sum_i L_i^2/2I_i$. The level surfaces of energy are concentric ellipsoids while those of L^2 are concentric spheres. Trajectories must lie on an intersection an ellipsoid and a sphere determined by initial conditions. Generally, the inertia ellipsoid and angular momentum sphere intersect along a pair if closed curves, each of which is a trajectory (corresponding roughly, to clockwise and counterclockwise rotation of the body).

While having conserved quantities helps in geometrically constraining the motion, they can be particularly helpful in solving the equations of motion if they generate 'commuting flows' on phase space. In Hamiltonian mechanics where the space of functions on phase space is equipped with a Poisson bracket, this leads to the notion of Liouville integrability. In essence, a Hamiltonian system is Liouville integrable if one can find a transformation from the original variables to action-angle variables on phase space.

For a system with n degrees of freedom and phase space $\mathbb{R}^{2n}$, action-angle variables (I_j, θ^j) for $1 \leq j \leq n$ are canonical coordinates $(\{I_i, I_j\} = \{\theta^i, \theta^j\} = 0, \{\theta^i, I_j\} = \delta^i_j)$ on $\mathbb{R}^{2n}$ such that the Hamiltonian is a function only of the action variables $H = H(I)$. Thus all the angle variables θ^i are cyclic and their conjugate momenta must be conserved. Indeed, Hamilton's equations $\dot{I}_j = -\frac{\partial H}{\partial \theta^j} = 0$ imply that the action variables are constants of motion and that the angle variables evolve linearly in time

$$\dot{\theta}^j = \frac{\partial H}{\partial I_j} = \Omega^j(I) \quad \Rightarrow \quad \theta^j(t) = \theta^j(0) + \Omega^j t. \tag{1}$$

Thus, if we have expressed our system in action-angle variables, then Hamilton's equations are easily integrated. Needless to say, this is possible only for a few very special systems.

[2] We may use the integrating factor $\dot{q}$ to reduce the order from second to first. Indeed, multiplying by $\dot{q}$ we get $m\ddot{q}\dot{q} = -V'(q)\dot{q}$ both sides of which are perfect differentials, so that $\frac{d}{dt}(\frac{1}{2}m\dot{q}^2 + V(q)) = 0$ leading to the conservation of energy.

[3] For a quadratic polynomial V, this integral can be expressed in terms of inverse trigonometric functions while for cubic and quartic polynomials, it can be expressed in terms of elliptic integrals. Inverting these lead to formulae for q that are trigonometric and (say) Jacobian elliptic functions of time. For more general potentials, the integrals may not be possible to evaluate in terms of known functions, but we will still regard the problem as solved since it has been reduced to quadrature (evaluation of integrals), which could be done, say numerically.

A theorem of Liouville and Arnold (which we will discuss in more detail later) tells us that this is possible provided the system possesses n conserved quantities $F_1, \cdots, F_n$ in involution (i.e., which Poisson commute $\{F_i, F_j\} = 0$ for all $1 \leq i, j \leq n$). We will see that under suitable hypotheses, the common level sets of the F_i are a family of n-dimensional 'invariant tori' (Cartesian products of n circles: $S^1 \times S^1 \times \cdots \times S^1$). Each trajectory is confined to one of these tori (determined by initial conditions) and is periodic or quasi-periodic according as the frequencies Ω^j are commensurate or not. Such a system is often said to be **completely integrable**.

• It is notable that Liouville integrability requires only n conserved quantities for us to be able to solve the equations of motion. This is much less than the $2n-1$ conserved quantities that are required to determine the trajectory purely through the geometric constraints they impose. Evidently, the requirement of being in involution (or generating commuting flows) is a strong condition which enables us to solve the equations of motion.

Harmonic Oscillator: The 1d harmonic oscillator is perhaps the simplest example of an integrable system. Newton's equation $\ddot{q} = -\omega^2 q$ and Hamilton's equations $\dot{q} = p/m$ and $\dot{p} = -kq$ follow from the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$ which is the only independent conserved quantity, with $\omega = \sqrt{\frac{k}{m}}$ and canonical PBs $\{q, p\} = 1$. Phase space trajectories are ellipses traversed clockwise:

$$q(t) = \sqrt{\frac{2E}{m\omega^2}} \sin\theta(t) \quad \text{and} \quad p(t) = \sqrt{2mE}\cos\theta(t) \quad \text{where} \quad \theta(t) = \omega(t - t_0). \tag{2}$$

Since $\theta = \arctan(m\omega q/p)$ evolves linearly in time, it is a natural choice for an angle variable (it is the clockwise angle the phase point makes with the p-axis). The corresponding action variable must be conserved, so it must depend on q and p only via the conserved energy H. It can be found by requiring $\{\theta, I\} = 1$ which leads to $I'(H) = 1/\omega$ (show this!). Thus, a pair of angle-action variables for the oscillator are

$$\theta(q,p) = \arctan\left(\frac{m\omega q}{p}\right) \quad \text{and} \quad I(q,p) = \frac{H(q,p)}{\omega} = \frac{p^2}{2m\omega} + \frac{1}{2}m\omega q^2 \tag{3}$$

Evidently, the Hamiltonian $H(\theta, I) = \omega I$ is a function of I alone, and θ is cyclic. $I(H)$ has dimensions of action and is geometrically $(1/2\pi)\times$ the area enclosed by the phase trajectory during one oscillation: the ellipse has semi-axes $A = \sqrt{\frac{2E}{k}}$ and $B = \sqrt{2mE}$ and area πAB:

$$I(H) = \frac{1}{2\pi}\oint p\, dq. \tag{4}$$

It had to be this way: The action-angle map $(q, p) \mapsto (\theta, I)$ is a canonical transformation (CT) since both pairs are canonical variables. CTs preserve areas, so the area enclosed by a trajectory is the same

$$\oint p dq = \oint I d\theta = I \times 2\pi \tag{5}$$

by the constancy of I. Angle-action variables are not unique. For instance, we may add a constant to I or θ without affecting their PBs nor the conservation of I. We may also rescale the action variable I_i by constant λ_i and the angle variable θ^i by $1/\lambda_i$ while retaining their status as action-angle variables.

Separation of variables: There are other notions of integrability even within Hamiltonian mechanics. One of them is separation of variables, which can be formulated via the Hamilton-Jacobi equation. Recall

that the time dependent HJ equation for a time-independent Hamiltonian is the first order nonlinear PDE

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q^i}) = 0, \quad \text{where} \quad p_i = \frac{\partial S}{\partial q^i}, \tag{6}$$

for Hamilton's principal function $S(q, P, t)$. For a particle in a potential it is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\frac{\partial S}{\partial q^i}\frac{\partial S}{\partial q^i} + V(q) = 0. \tag{7}$$

When S exists, it generates a canonical transformation from the original variables to new coordinates and momenta which are both constant in time (so that the new Hamiltonian can be taken to vanish). The HJ equation may regarded as the semi-classical limit (leading order for small $\hbar$) of the time-dependent Schrödinger equation upon making the ansatz $\psi = e^{iS/\hbar}$. As for energy eigenstates $\Psi(q,t) = e^{-iEt/\hbar}\psi(q)$ in QM we can separate time time dependence by separation of variables $S(q,t) = -Et + W(q)$. $W(q)$ is called Hamilton's characteristic function and must satisfy the time-independent HJ equation $E = H(q, \partial_i W(q))$.

For a system with n degrees of freedom, separation of variables involves finding a coordinate system where Hamilton's characteristic function $W(q) = \sum_i W_i(q_i)$ can be written as a sum of n functions each of which depends on only one of the n coordinates. This is the classical analogue of a product form for the QM wave function when we, say, separate radial from angular dependence in the hydrogen energy eigenfunctions. Such a separation of variables is possible for the isotropic harmonic oscillator (in Cartesian coordinates) or Kepler problem (in spherical polar coordinates) or Euler 3 body problem (in elliptical coordinates). In this process, n constants of integration arise which may be taken as the conserved new momenta P_j that appear in the generating function $S(q^i, P_j)$ for a canonical transformation of the second type.

Superintegrable systems A system with n degrees of freedom is said to be superintegrable if it has more than n independent conserved quantities. It is maximally superintegrable if it has $2n-1$ independent conserved quantities. Note that even if there are more than n independent constants of motion, at most n of them can be in involution. The Kepler problem $V = -\alpha/r$ is maximally superintegrable: in center of mass variables, it has 3 degrees of freedom and five independent conserved quantities: energy E and the components of the angular momentum and Laplace-Runge-Lenz (LRL) vectors $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{A} = \mathbf{p} \times \mathbf{l} - m\alpha\hat{r}$ subject to the two relations $\mathbf{A} \cdot \mathbf{l} = 0$ and $A^2 = 2mEl^2 + m^2\alpha^2$. For a superintegrable system, the HJ equation is separable in more than one coordinate system: E.g., spherical polar and parabolic cylinder coordinates for the Kepler problem. Similarly, the free particle HJ equation can be separated in cartesian, spherical polar, elliptic cylinder, parabolic cylinder coordinates etc. The quantum manifestation of this is the possibility of separating variables via a product form for the wavefunction in more than one coordinate system in order to solve the time-independent Schrödinger equation.

2.2 Liouville-Arnold Theorem

Recall that if we have expressed our system in action-angle variables, then Hamilton's equations are easily integrated. The Liouville-Arnold theorem [15] provides sufficient conditions for finding a canonical transformation (CT) from a given set of canonical phase space variables (q^i, p_j) to action-angle variables. Essentially, one needs to have n conserved quantities that are in involution (i.e., generate 'commuting flows').

Liouville-Arnold Theorem: Consider a system with Hamiltonian H and phase space $\mathbb{R}^{2n}$ with coordinates q^i and momenta p_j satisfying canonical PBs

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = 0 \quad \text{and} \quad \{p_i, p_j\} = 0. \tag{8}$$

Now suppose the system admits n independent conserved quantities $F_i(q,p)$ with the Hamiltonian being expressible as a function of them. Moreover, suppose the conserved quantities are in involution (i.e., Poisson commute):

$$\dot{F}_i = 0 \quad \text{and} \quad \{F_i, F_j\} = 0 \quad \text{for} \quad i,j = 1,2,\ldots,n, \tag{9}$$

and that their common level sets $M_f = \{(q,p) | F_i = f_i \text{ for } \ 1 \leq i \leq n\} \subset \mathbb{R}^{2n}$ foliate the phase space. These common level sets are of course invariant under time evolution. Suppose further that M_f is compact and connected, then it may be shown to be diffeomorphic to an n-dimensional 'invariant' torus T^n [6]. Under these hypotheses, there is a CT from (q^i, p_j) to angle-action variables (θ^i, I_j) with the property that the Hamiltonian is a function of the action variables alone, i.e., $H = H(I_1, I_2, \cdots, I_n)$.

Comments and Sketch of the Proof: A system that satisfies the hypotheses of the LA theorem is said to be Liouville integrable or completely integrable. Though the F_i are conserved and in involution, they are not in general a natural choice for the action variables since it is not clear how to find canonically conjugate angle variables that evolve linearly in time. On the other hand, the harmonic oscillator with $n = 1$ suggests how one may proceed. In this case, energy $E = p^2/2m + m\omega^2 x^2/2$ is the only conserved quantity and $F_1 = E$. Also, $E = f_1$ is an ellipse (1d torus T^1) and the action variable is defined as

$$I = \frac{1}{2\pi} \oint_{T_1} p dq, \quad \text{which turns out to equal} \quad \frac{E}{\omega}. \tag{10}$$

Generalizing this to n degrees of freedom, M_f is an n-dimensional torus. So it has n independent non-contractible cycles Γ_k for $k = 1, 2, \cdots, n$. It is thus natural to define

$$I_k = \frac{1}{2\pi} \oint_{\Gamma_k} p_i dq^i. \tag{11}$$

Here, the momenta p_i are regarded as functions of the q's and f's. Indeed, assuming that the matrix of first partials $\frac{\partial F_j}{\partial p_k}$ is non-singular (non-zero determinant), we may express $p_i = p_i(q,f)$ by solving the equations $F_j(p,q) = f_j$. Now, we will show that I_k depend only on the values of the conserved quantities F and are unchanged under continuous deformation of the closed curves Γ_k. In particular, the line integral of $\alpha = p_i dq^i$ around a contractible closed contour on M_f vanishes. To show this, differentiate $F_i(q, p(q,f)) = f_i$ in q^j and contract with $\frac{\partial F_m}{\partial p_j}$:

$$\frac{\partial F_i}{\partial q^j} + \sum_k \frac{\partial F_i}{\partial p_k}\frac{\partial p_k}{\partial q^j} = 0 \quad \forall \ i,j \quad \Rightarrow \quad \sum_j \frac{\partial F_m}{\partial p_j}\frac{\partial F_i}{\partial q^j} + \sum_{j,k} \frac{\partial F_m}{\partial p_j}\frac{\partial F_i}{\partial p_k}\frac{\partial p_k}{\partial q^j} = 0 \quad \forall \ i,m. \tag{12}$$

Anti-symmetrizing in the indices i and m gives

$$\{F_i, F_m\} + \sum_{j,k} \left(\frac{\partial F_m}{\partial p_j}\frac{\partial F_i}{\partial p_k}\frac{\partial p_k}{\partial q^j} - \frac{\partial F_i}{\partial p_j}\frac{\partial F_m}{\partial p_k}\frac{\partial p_k}{\partial q^j} \right) = 0. \tag{13}$$

The PB vanishes as the F's are in involution. Relabeling indices and using invertibility of $\frac{\partial F_i}{\partial p_k}$, we get

$$\sum_{j,k} \frac{\partial F_i}{\partial p_k}\frac{\partial F_m}{\partial p_j}\left(\frac{\partial p_k}{\partial q^j} - \frac{\partial p_j}{\partial q^k} \right) = 0 \quad \text{or} \quad \frac{\partial p_k}{\partial q^j} - \frac{\partial p_j}{\partial q^k} = 0. \tag{14}$$

The latter condition says that if we view p_k as the components of the 'canonical' 1-form $\alpha = p_k dq^k$ on the torus M_f, then it is closed $d\alpha = 0$. Stokes' theorem $\int_S d\alpha = \oint_{\partial S} \alpha$ [where S is a surface whose boundary is ∂S] then implies that $\oint p_j dq^j$ vanishes around any contractible closed curve on the torus. More generally, if $\tilde{\Gamma}_k$ is a curve obtained by continuously deforming the cycle Γ_k and S is the strip whose boundary is the union of Γ_k and $\tilde{\Gamma}_k$ traversed backwards, then by Stokes' theorem,

$$\oint_{\Gamma_k} p_j dq^j - \oint_{\tilde{\Gamma}_k} p_j dq^j = \int_S \left(\frac{\partial p_i}{\partial q^j} - \frac{\partial p_j}{\partial q^i} \right) dq^j \wedge dq^i = 0. \tag{15}$$

Thus, I_k are well-defined functions of the F's: they are unchanged under continuous deformations of the n independent cycles Γ_k. As a consequence, they are conserved. What is more, they are in involution:

$$\{I_i, I_j\} = \sum_{r,s,k} \frac{\partial I_i}{\partial F_r} \frac{\partial F_r}{\partial q^k} \frac{\partial I_j}{\partial F_s} \frac{\partial F_s}{\partial p_k} - \frac{\partial I_i}{\partial F_r} \frac{\partial F_r}{\partial p_k} \frac{\partial I_j}{\partial F_s} \frac{\partial F_s}{\partial q^k} = \sum_{r,s} \frac{\partial I_i}{\partial F_r} \frac{\partial I_j}{\partial F_s} \{F_r, F_s\} = 0. \tag{16}$$

Thus I_k are candidate action variables. Can we find canonically conjugate variables that qualify as angle variables? We will define a set of variables θ^k conjugate to I_k by specifying a canonical transformation from $(q, p) \to \theta, I$. Inspired by what works for the harmonic oscillator, we pick an invariant torus defined by the values f_i of the constants F_i and consider a generating function of the 'second' kind

$$S(q, I) = \int_{q_0}^{q} p_j(q, f) dq^j. \tag{17}$$

By changing the invariant torus, S is defined on the whole phase space. The integral is along a curve on M_f from q_0 to q. Here, we regard p_j as functions of the q's and the f's (which may in turn be regarded as functions of the I's) obtained by solving $F_i(q, p) = f_i$. We have already shown that the canonical 1-form $\alpha = p_j dq^j$ is closed when restricted to M_f. This implies that $S = \int_{q_0}^{q} \alpha$ is invariant under continuous deformations of the path from q_0 to q. Thus, S may be regarded as a function of the endpoints. Changing the base point $q_0 \to q_0'$ will add to S a 'constant' $\int_{q_0}^{q_0'} pdq$ that depends on the I's. The equations of transformation for such a generating function of the second kind express the new coordinates and old momenta as

$$\theta^i = \frac{\partial S}{\partial I_i}, \quad \text{and} \quad p_i = \frac{\partial S}{\partial q^i}. \tag{18}$$

The proposed angle variables θ^i depend on the base point, so they are far from unique. Having a generating function ensures that the transformation is canonical[4] i.e., it preserves the PBs of the q^i and p_j:

$$\{\theta^j, \theta^k\} = 0, \quad \{I_j, I_k\} = 0 \quad \text{and} \quad \{\theta^j, I_k\} = \delta^j_k. \tag{19}$$

Moreover, since H was a function of the F's, it is a function of the action variables alone, $H = H(I_1, \cdots, I_n)$. In action-angle variables, Hamilton's equations become

$$\dot{I}_i = -\frac{\partial H}{\partial \theta^i} = 0 \quad \text{and} \quad \dot{\theta}^i = \frac{\partial H}{\partial I_i} \equiv \Omega^i(I). \tag{20}$$

Thus, the θ^i evolve linearly in time $\theta_k(t) = \Omega_k(I)t + \theta_k(0)$ while the action variables are conserved. The former generalize the variable $\theta = \arctan(m\omega q/p)$ of the harmonic oscillator, which is the angle

[4] To see this, we compute the exterior derivative $dS(q, I) = \frac{\partial S}{\partial q^i} dq^i + \frac{\partial S}{\partial I_i} dI_i = p_i dq^i + \theta^i dI_i$. Taking another exterior derivative, $d^2 S = 0$ or in other words, $\sum_i dp_i \wedge dq^i + \sum_j d\theta^j \wedge dI_j = 0$. Thus, the transformation preserves the form of the symplectic form $\omega = dq^i \wedge dp_i = d\theta^j \wedge dI_j$ which means it is a symplectic diffeomorphism or canonical transformation.

around an invariant ellipse $(1/2)(p^2/m + m\omega^2 x^2) = E$. In fact, we will show that $\oint_{\Gamma_k} d\theta^i = 2\pi\delta^i_k$, so that θ^k may be viewed as the angle around the k^{th} cycle of an invariant torus. Indeed, using (18) and observing that $dI_j = 0$ on M_f,

$$\oint_{\Gamma_k} d\theta^i = \frac{\partial}{\partial I_i} \oint_{\Gamma_k} dS(q, I) = \frac{\partial}{\partial I_i} \oint_{\Gamma_k} \left(p_i dq^i + \theta^j dI_j\right) = 2\pi \frac{\partial I_k}{\partial I_i} = 2\pi\delta^i_k \tag{21}$$

Remark: The frequencies Ω^i vary from torus to torus, as they depend on the values of F_i or I_i. If the frequencies Ω^i are commensurate (Ω^i/Ω^j rational for all choices of i, j), then the trajectory on that invariant torus is closed and the motion periodic. Generically, the frequencies are not commensurate and the trajectories on an invariant torus are not periodic, but rather quasi-periodic: a superposition of several incommensurate periodic motions. The corresponding trajectory is not closed but dense on the torus: if we wait sufficiently long, it will come arbitrarily close to any given point on the torus without ever crossing itself (draw a picture for 2 degrees of freedom). In this case, the dynamics is ergodic (time and ensemble averages agree) on the torus (though not on the full phase space). However, the time evolution is predictable and certainly not chaotic.

2.3 Remarks on commuting flows

Remark on commuting flows: Since the conserved quantities F_j are in involution, they generate n commuting flows. What does this mean? Each F_j generates a flow with a corresponding time t_j. The evolution of any observable, g is given by $\frac{dg}{dt_n} = \{g, F_n\}$. Since $\{F_i, F_j\} = 0$, all the F_js are conserved quantities for each of the flows. Thus, by sequentially choosing $F_1, F_2, \cdots$ as the Hamiltonian, we get a hierarchy of integrable systems.

Remark on Frobenius Theorem for vector fields [45]: Suppose M is a smooth manifold and $V_{(1)}, \cdots V_{(m)}$ are a set of m vector fields on a region $U \subset M$ that form a subalgebra of the space of all vector fields on U, i.e., $[V_{(i)}, V_{(j)}] = \sum_k c^k_{ij} V_{(k)}$ for some functions c^k_{ij} on U. Then, the integral curves of these vector fields mesh together to form a family of submanifolds of M. The dimension of each submanifold is the dimension of the vector space spanned by the vector fields and is at most m. Each point of U lies on precisely one such submanifold, provided the dimension of span $(V_{(1)}, \cdots V_{(m)})$ is the same everywhere. The submanifolds fill up (foliate) U and each submanifold is called a leaf of the foliation. In the context of the Liouville-Arnold theorem, M is the $2n$ dimensional phase space, the vector fields V_{F_i} are the canonical/Hamiltonian vector fields associated to the n conserved quantities $F_1, \cdots, F_n$. Since F_i are in involution, the corresponding canonical vector fields commute. Their integral curves mesh together to form a family of n dimensional submanifolds (the invariant tori M_f) which foliate M.

2.4 Integrability in field theory

For classical fields, notions of integrability are even more varied. A brief historical survey of the first integrable field theory (Korteweg-de Vries or KdV) will be given later, it is a good way of beginning to understand the development of the subject. Here, we list some features of integrable field theories.

2.5 Some features of integrable classical field theories

Not all field theories that we would like to designate as integrable are integrable in the same sense. Some of the features such models may possess include:

1. Solitary waves (n soliton solutions) displaying 'soliton' scattering: solitary waves retain their shape after interacting upto a shift in phase. Scattering phase shift factorizes as a sum of pair scattering phase shifts.
2. Infinitely many conserved quantities in involution
3. Lax pair or zero curvature representation to generate infinitely many conserved quantities.
4. Classical r-matrix to ensure the conserved quantities are in involution.
5. Bäcklund transformation to generate new solutions from existing ones.
6. Presence of lots of (infinitely many) symmetries (beyond Noether symmetries).
7. Prolongation algebra.
8. Bi-Hamiltonian structure or Poisson pencil.
9. Inverse scattering transform or linearizing map like that to action-angle variables. Plays role similar to Fourier transform for linear PDEs.
10. Infinitely many explicit solutions.
11. Tau function and Hirota bilinear form.

This is of course an incomplete list, there are many more viewpoints and features of integrable field theories.

2.6 Integrability of quantum systems

For quantum systems, integrability usually refers to a way of finding the spectrum of the Hamiltonian, the energy eigenfunctions as well as the scattering matrix, if there is a scattering interpretation.

Liouville integrable classical Hamiltonian systems that admit a quantization where the Poisson commuting integrals of motion are represented as commuting operators on a Hilbert space are perhaps the simplest examples of quantum mechanical integrable systems.

The quantum free particle, particle in an infinite well, delta function potential, harmonic oscillator, isotropic rigid body and hydrogen atom are among the simplest examples of solvable quantum systems. There are many special potentials in 1d for which the spectrum can be found exactly (e.g. using methods of supersymmetric quantum mechanics).

A significant step from one- or two-body to interacting many-body quantum systems was achieved by Hans Bethe in 1931, who was able to diagonalize the Hamiltonian for a linear chain model of a 1d magnet (anti-ferromagnetic Heisenberg model) via the ansatz that bears his name. The Bethe ansatz has since been extended to many other 1d many body quantum systems including the Bose gas with contact interactions (Lieb-Liniger model). In models solvable by the Bethe ansatz, the many particle scattering-matrix factorizes as a product of 2-particle S-matrices and many body wavefunctions may be constructed from two-body wavefunctions.

More generally, a quantum system that allows for scattering is regarded as integrable if the number of particles as well as the set of incoming momenta are conserved[5]. This is typically associated with factorized scattering so that many-particle collisions may be broken down into independent pair collisions [10]. In these cases, one says that the S-matrix displays soliton behavior. This situation may be viewed as the quantum analogue of the factorization of scattering phase shifts as a sum of pair-scattering phase shifts in classical n-soliton scattering. In the quantum theory, the equality of different ways in which multi-particle scattering can be factorized into pair scatterings leads to the celebrated Yang-Baxter equation.

Another class of integrable quantum systems are those that can be reduced to models of free fermions.

2.7 Examples of integrable systems

• **Mechanical systems:** Linear harmonic oscillator, Kepler problem, Euler 3-body problem (mass moving under gravitational force of two fixed masses), Rigid bodies: Euler (force free), Lagrange (heavy symmetrical top with center of gravity on symmetry axis) and Kovalevskaya (heavy symmetrical top with $I_1 = I_2 = 2I_3$) tops, Jacobi's geodesic motion on a triaxial ellipsoid, Neumann model (particle on a sphere connected by springs to coordinate planes), Kirchhoff model (rigid body moving in an ideal fluid), Rajeev-Ranken model (like coupled tops, a reduction of a 1+1 d field theory) etc.

• **One-dimensional n-body problems:** Toda chain (exponential nearest neighbor potentials), n-vortex problem (PK Newton), Calogero-Moser model (rational $1/x^2$, trigonometric $k^2/\sin^2(kx)$, hyperbolic $k^2/\sinh^2(kx)$ and elliptic $\wp(x,\omega_1,\omega_2)$ potentials between all pairs of masses). These pair potentials have a Lie algebraic interpretation in terms of the $\mathrm{su}(n+1)$ Lie algebra; there are generalizations to other simple Lie algebras. The Ruissenaars-Schneider model is a relativistic generalization of Calogero. Relativistic generalizations of Toda and Calogero are known for all simple Lie algebras.

• **Lattice statistical mechanical models:** 1d and 2d Ising model (Onsager), 6 vertex model (Lieb, Baxter), 8 vertex model (Baxter), hard hexagon model (Baxter), chiral Potts model (Baxter).

• **Quantum spin chains:** spin-$\frac{1}{2}$ isotropic Heisenberg (XXX) spin chain, spin-s XXX chain, XXZ chain, XY spin chain, XYZ spin chain.

• **Difference-differential equations:** where we discretize the space variable. Integrable discretizations of soliton equations, discrete equations associated with Bäcklund transformations of soliton equations. ABS (Adler-Bobenko-Suris) classification includes 'lattice equations' such as discrete KdV, Nijhoff, Quispel and Capel equations [43].

• **Integrable field theories:**

1. **1+1 dimensions:** KdV and mKdV equations, nonlinear Schrödinger equation (NLSE), Heisenberg magnetic chain, Vortex filament equation, free scalar and fermion fields, massless Thirring model and Federbush model (relativistic, no solitons but solvable by linearizing map), sine-Gordon model, massive Thirring model, Gross-Neveu model, SU(2) principal chiral model, nonlinear sigma models, Kondo problem, various conformal field theories, Liouville field theory, sinh-Gordon model, Toda field theory, $AdS_5 \times S^5$ string sigma model (see [8] for a review).

2. **2+1 dimensions:** Kadomtsev-Petviashvili (KP) equation for surface waves (a generalization of KdV), Davey-Stewartson equation for water waves (a generalization of NLSE).

[5]Note that crossing symmetry in integrable quantum field theories relates s-channel to t-channel scattering rather than $2 \to 2$ scattering to, say, $1 \to 3$ decays

3. **3+1 dimensions:** Euclidean (anti)-self-dual Yang-Mills equations (has Lax pair and ∞ of explicit instanton solutions), Ernst equations in general relativity (axisymmetric spacetimes are the 'soliton' solutions), $\mathcal{N} = 4$ super Yang-Mills.

Interconnections among integrable models: There are many interrelationships among integrable models. Some integrable mechanical systems arise as reductions of (integrable) field theories. Many integrable field theories arise as reductions of the anti-self-dual-Yang-Mills equations. The Ruissenaars-Schneider model describes interactions among relativistic point particles that have the same scattering as sine-Gordon solitons. The Rajeev-Ranken model is a mechanical reduction of a dual to the principal chiral model. Some integrable spin chains arise as limits of $\mathcal{N} = 4$ super Yang-Mills theory [8]. Some integrable systems arise as reductions of free particle motion on larger spaces.

2.8 Integrable systems in the wider physical context

Why should we care about integrable systems? Most mathematical models for physical systems are not integrable. Integrable systems are rare or non-generic in the sense that a typical perturbation to an integrable Hamiltonian will render it non-integrable. Phase trajectories of integrable systems are in general quasi-periodic in time and do not display chaotic behavior. Nevertheless, several integrable models are physically realized and have played an important role: e.g. Kepler problem, top, Hydrogen-atom, sine-Gordon model (model for solitons in particle physics, other applications), nonlinear Schrödinger (nonlinear optics, Bose gases), KdV (water waves), etc. Moreover, we understand many non-integrable systems by treating them as perturbations of integrable systems. What is more, due to their special features, we can often say more about an integrable system (and with greater confidence), so that they can serve as theoretical laboratories for testing physical concepts and techniques and to provide proofs of principle for physical mechanisms (such as the possibility of a mass gap from studying the 1+1d SU(2) principal chiral model or holography by studying $\mathcal{N} = 4$ super Yang-Mills in 3+1d and the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string sigma model).

3 Lax pairs and conserved quantities

3.1 Lax pair for the harmonic oscillator

We will use the harmonic oscillator to provide an illustration of the idea of a Lax pair. The idea of a Lax pair is based on the observation that the equations of motion

$$\dot{x} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -m\omega^2 x. \tag{22}$$

are equivalent to the *Lax equation* $\dot{L} = [L, A]$ for the pair of 2×2 matrices [7]

$$L = \begin{pmatrix} p/m & \omega x \\ \omega x & -p/m \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & \omega/2 \\ -\omega/2 & 0 \end{pmatrix}, \tag{23}$$

whose entries can depend on the dynamical variables x and p. Indeed, one verifies that the four equations below coincide with (22):

$$\dot{L} = \begin{pmatrix} \dot{p}/m & \omega\dot{x} \\ \omega\dot{x} & -\dot{p}/m \end{pmatrix} = [L, A] = \begin{pmatrix} -\omega^2/x & p\omega/m \\ p\omega/m & \omega^2/x \end{pmatrix}. \tag{24}$$

We notice the following feature of the Lax matrix, $\mathrm{tr}\, L^2 = 2(p^2/m^2 + \omega^2 x^2)$ is $(m/4)$ times the conserved energy of the harmonic oscillator. It turns out that this is a general feature: one may use the Lax matrix to obtain conserved quantities.

Remark: A Lax pair for a given system of equations (if it exists) is not unique. For instance, we may add to A a matrix that commutes with L and add to L a time independent matrix that commutes with A without altering the Lax equation $L_t = [L, A]$. Systems can even possess Lax pairs of different dimensions.

3.2 Isospectral evolution of the Lax matrix

The Lax equation ensures that the eigenvalues (spectrum) of L are independent of time. This property is known as isospectrality[6]. To understand this, let us consider the Lax equation

$$L_t \equiv \dot{L} = [L, A], \tag{25}$$

where L and A are matrices with entries depending on the dynamical variables[7]. We have used subscripts to denote derivatives. Since the trace of the commutator of a pair of finite dimensional matrices vanishes, $\mathrm{tr}\, L$ is independent of time. In fact, we can also show that $\mathrm{tr}\, L^2$ is independent of time. Indeed,

$$\partial_t L^2 = LL_t + L_t L = L[L, A] + [L, A]L = L^2 A - LAL + LAL - AL^2 = [L^2, A]. \tag{26}$$

Since this is a commutator, we have $\partial_t \,\mathrm{tr}\, L^2 = 0$. Similarly, one checks that $\partial_t L^3 = [L^3, A]$. It is then possible to show by induction that $\partial_t L^n = [L^n, A]$. Thus $\mathrm{tr}\, L^n$ is conserved for any $n = 1, 2, 3, \cdots$.

• For a matrix, $\mathrm{tr}\, L^n = \sum_j \lambda_j^n$ is a sum of powers of eigenvalues of L and conversely λ_j can be expressed in terms of $\mathrm{tr}\, L^n$. Thus the conservation of $\mathrm{tr}\, A^n$ implies that the eigenvalues of L are conserved.

• For the harmonic oscillator (23), $\mathrm{tr}\, L = 0$, but $\mathrm{tr}\, L^2$ is nontrivial and a multiple of the conserved energy:

$$\mathrm{tr}\, L^2 = \frac{2}{m}\left(\frac{p^2}{m} + m\omega^2 x^2\right) = 4\frac{E}{m}. \tag{27}$$

Also note that for $E > 0$, L has two distinct eigenvalues $(\pm\sqrt{2E/m})$ leading to one-dimensional eigenspaces, as was assumed in Eqn. (32). Furthermore, for $n = 1, 2, 3, \ldots$,

$$L^{2n} = \left(\frac{p^2}{m^2} + x^2\omega^2\right)^n \mathbf{1} = \left(\frac{2E}{m}\right)^n \mathbf{1} \quad \text{and} \quad L^{2n+1} = \left(\frac{2E}{m}\right)^n L. \tag{28}$$

Thus, $\mathrm{tr}\, L^{2n} = 2\,(2E/m)^n$ while $\mathrm{tr}\, L^{2n+1} = 0$ so that the traces of higher powers of L do not furnish any new conserved quantities, which is forbidden for a system with one degree of freedom.

[6] Isospectral evolution means the spectrum of the Lax matrix is independent of time.

[7] We notice that the Lax equation bears a resemblance to the Heisenberg equation of motion for an operator F in a quantum system with Hamiltonian H: $i\hbar\dot{F} = [F, H]$. However, we cannot always take the trace of this equation and deduce that $\mathrm{tr}\, F$ is a useful conserved quantity. To begin with, $\mathrm{tr}\, F$ may simply be infinite. Moreover, the trace of a commutator of infinite dimensional operators (like the position or momentum or Hamiltonian of a particle) does not always vanish. The infinite sums involved in $\mathrm{tr}\, FH$ and $\mathrm{tr}\, HF$ typically diverge.

• There is another way to understand the isospectrality of L. We begin with the eigenvalue problem $L\psi = \lambda\psi$. Differentiating in time,

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{29}$$

Upon using the Lax equation (25) this becomes

$$(LA - AL)\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{30}$$

Utilizing $L\psi = \lambda\psi$ and rearranging, we get

$$(L-\lambda)A\psi + (L-\lambda)\psi_t = \lambda_t\psi \quad \text{or} \quad (L-\lambda)(\psi_t + A\psi) = \lambda_t\psi. \tag{31}$$

For the eigenvalue λ to be time-independent ($\lambda_t = 0$), the LHS must vanish. Thus, $\psi_t + A\psi$ must be an eigenfunction of L with eigenvalue λ. Recall that ψ too is an eigenstate of L with the same eigenvalue. Now, if the λ-eigenspace of L is one-dimensional (see below for other methods of showing the isospectrality of a Lax matrix without assuming that its eigenspaces are one-dimensional.), which implies that $\psi_t + A\psi$ must be a multiple of ψ:

$$\psi_t + A\psi = \beta\psi, \tag{32}$$

for some (possibly time-dependent) complex number β. This equation may be viewed as an evolution equation[8] for ψ:

$$\psi_t = (-A + \beta\mathbf{1})\psi. \tag{33}$$

Here, $\mathbf{1}$ is the identity matrix. Thus, the Lax equation $L_t = [L, A]$ and this evolution equation for ψ (which we could postulate even if the λ eigenspace of L is degenerate) together imply that the eigenvalue λ is a conserved quantity. We say that L evolves isospectrally.

• There are other ways to show that the eigenvalues of L are time-independent.

1. For instance, suppose we assume that L is hermitian so that λ is real. We take an inner product of $(L-\lambda)(\psi_t + A\psi) = \lambda_t\psi$ (see Eqn. (31)) with the eigenfunction ψ and use hermiticity to get

$$\langle (L-\lambda)(\psi_t + A\psi), \psi\rangle = \langle \lambda_t\psi, \psi\rangle \quad \text{or} \quad \langle (\psi_t + A\psi), (L-\lambda)\psi\rangle = \lambda_t\|\psi\|^2. \tag{34}$$

The LHS vanishes as $L\psi = \lambda\psi$. Moreover, being an eigenfunction, $||\psi|| \neq 0$, so we must have $\lambda_t = 0$.

2. The isospectrality of $L(t)$ may also be established by showing that $L(t)$ is similar to $L(0)$. Indeed, suppose we define the invertible matrix $S(t)$ via the equation $\dot{S} = -AS$ with the initial condition $S(0) = \mathbf{1}$, then the solution of the Lax equation with initial condition $L(0)$ is $L(t) = S(t)L(0)S^{-1}(t)$. This is easily verified:

$$\dot{L}(t) = \partial_t(SL(0)S^{-1}) = -ASL(0)S^{-1} - SL(0)S^{-1}\partial_t(S)S^{-1} = -AL(t) + L(t)A = [L(t), A]. \tag{35}$$

Here we used $\partial_t(SS^{-1}) = \partial_t\mathbf{1} = 0$, to write $\partial_t(S^{-1}) = -S^{-1}\partial_t(S)S^{-1}$. Finally, we observe that two matrices related by a similarity transformation have the same eigenvalues:

$$L(0)\psi = \lambda\psi \quad \Rightarrow \quad SL(0)S^{-1}(S\psi) = \lambda(S\psi) \quad \text{or} \quad L(t)(S\psi) = \lambda(S\psi). \tag{36}$$

Thus, the eigenvalues of L are conserved in time.

[8] It is possible to absorb the $\beta\mathbf{1}$ term into A since it commutes with L and therefore does not affect the Lax equation.

4 KdV equation and its integrability

4.1 Physical background and discovery of inverse scattereing

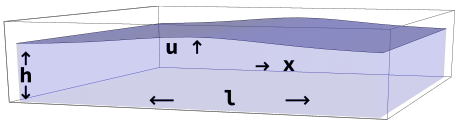

Figure 1: Surface wave profile in a canal.

In appropriate units, the KdV equation is the PDE

$$u_t - 6uu_x + u_{xxx} = 0 \tag{37}$$

introduced by Korteweg and de Vries in 1895. It was introduced to describe, the height u of long wavelength (shallow) surface waves in water flowing along a canal[9]. The KdV equation for the 'field' u describes the evolution of infinitely many degrees of freedom labeled by points x lengthwise along the canal. In a sense, it is a universal equation for weakly nonlinear dispersive waves in one spatial dimension and appears in many areas of science. The nonlinear uu_x is called the 'advection' term, it can steepen the gradient of the profile. The u_{xxx} term is dispersive, it tends to spread a wave out since different Fourier modes travel at different speeds. Indeed, if we ignore the nonlinear term, we get a cubic dispersion relation $\omega = -k^3$ for a monochromatic wave with harmonic time dependance $u = e^{i(kx-\omega t)}$. In fact, the nonlinear and dispersive effects can balance each other leading to an exact solution $u = -\frac{1}{2}c\,\mathrm{sech}^2((\sqrt{c}/2)(x - ct - x_0))$ which describes a solitary wave of permanent form that travels at speed c. The presence of such a stable solitary wave solution made the KdV equation an attractive equation for modeling the wave of translation observed by Scott Russell in the Edinburgh-Glasgow canal in 1834. The KdV equation lay largely dormant during the first half of the 20^{th} century until 1955 when Fermi, Pasta, Ulam and Tsingou numerically studied a model for vibrations (phonons) in an anharmonic 1d lattice of atoms connected by anharmonic springs. The FPUT model turns out to be a discretization of the KdV equation. FPUT had expected the initial energy in the lattice to get equally distributed among the various modes of oscillation and the system to reach thermal equilibrium. To their surprise, this did not happen. Rather, the energy returned almost entirely to the initially excited mode and the dynamics was recurrent. Inspired in part by this, Kruskal and Zabusky [51] used the KdV equation to numerically model ion acoustic waves in a collisionless plasma and discovered the recurrence of initial states. What is more, they found that an initial profile $u(x)$ that in the absence of the dispersive term would evolve into a singularity (gradient catastrophe or shock where u_x diverges) instead developed a train of solitary of waves in the presence of the dispersive term. Remarkably, they found that these solitary waves could collide with each other and asymptotically re-emerge while retaining their shapes, just like hadrons and other particles in particle physics. The coined the term soliton for such solitary waves. The effect of the nonlinearity was to introduce a shift in the phases of the solitary waves at late time compared to their phases if they were not to have interacted. The observation of recurrence (lack of thermalization) and soliton scattering suggested that the KdV equation had more conserved

[9]The KdV equation describes long wavelength ($l \gg h$, 'shallow-water') surface waves of elevation $u(x,t) \ll h$ in water flowing in a narrow canal of depth h (see Fig. 2).

quantities than the obvious ones like mass, momentum and energy[10]. In 1967-68, through a series of heroic calculations, Whitham, and then Kruskal and Zabusky discovered a fourth and fifth conserved quantity[12]. Miura discovered yet more and there were 8 additional conserved quantities.

It was then shown through the work of Gardner, Greene, Kruskal and Miura (GGKM) that KdV admitted infinitely many independent constants of motion. They did not stop there. In a series of papers from 1967-1974, Gardner, Greene, Kruskal, Miura and Su developed a remarkable analytical method called the Inverse Scattering Transform to solve the KdV initial value problem on the real line. This method, which has since been extended to many other special PDEs (nonlinear Schrödinger, sine-Gordon etc) may be regarded as a vast nonlinear generalization of the Fourier transform technique of solving linear PDEs with constant coefficients, such as the wave and heat equations. Very roughly, the method proceeds by first finding and studying an auxiliary linear system of equations followed by solving a linear integral equation. For KdV, the auxiliary linear system includes (coincidentally) the eigenvalue problem for the Schrödinger operator in a potential given by the initial wave profile $u(x,0)$. Interestingly, though u evolves in a nonlinear fashion, GGKM showed that the scattering data (reflection and transmission amplitudes, discrete spectrum and wavefunction normalization) evolve in a very simple manner in time. Thus, one solves the direct scattering problem in the potential $u(x,0)$ and determines the scattering data at $t = 0$ and evolves them forward to time t via the GGKM equations. Even more remarkably, it is possible to recover the wave profile $u(x,t)$ from the scattering data at time t by solving the GLM linear integral equation derived by Gelfand and Levitan [24] and Marchenko [34] even earlier. Thus, by solving certain linear problems, one obtains the solution to the IVP for the nonlinear KdV equation. The last step of inverting the direct scattering problem gives the method the name *inverse scattering transform*.

Interestingly, part of the scattering data: the discrete spectrum and the transmission amplitude turn out to be independent of time. They thus lead to conserved quantities for the KdV equation. Moreover, the simple time evolution of scattering data allows one to discover a set of action-angle variables for the KdV equation, thus establishing it as a completely integrable infinite-dimensional Hamiltonian system.

The KdV equation is in a sense the simplest/minimal 1d first order nonlinear dispersive wave equation. Recall that the linear d'Alembert wave equation $u_{tt} = c^2 u_{xx}$ describes bidirectional propagation. Factorizing the wave operator we get 2 first order equations $u_t \pm c u_x = 0$ which describe unidirectional (right or leftward) propagation. The simplest source of nonlinearity is to replace the constant speed with u: $c \to u$. This leads to the kinematic wave equation of Hopf equation or inviscid Burgers equation, say $u_t + u u_x = 0$. $u u_x$ is the standard advection term of Eulerian hydrodyanamics. This equation describes

[10] Integrating (37) gives

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = \int_{-\infty}^{\infty} \partial_x \left(3u^2 - u_{xx}\right)\, dx = 0, \tag{38}$$

assuming $u \to 0$ as $x \to \pm\infty$. This leads to the conservation of the mean height $M = \int_{-\infty}^{\infty} u\, dx$. Furthermore, one may check by differentiating in time and using (37) that

$$P = \int_{-\infty}^{\infty} u^2 dx \quad \text{and} \quad E = \int_{-\infty}^{\infty} \left(u^3 + \frac{u_x^2}{2}\right) dx \tag{39}$$

are also conserved[11]. P and E can be interpreted as the momentum and energy of the wave.

[12] The first two of these additional integrals of motion are

$$\begin{aligned} Q_1 &= \int_{-\infty}^{\infty} \left(u^4 + 2u u_x^2 + \frac{1}{5} u_{xx}^2\right) dx \quad \text{and} \\ Q_2 &= \int_{-\infty}^{\infty} \left(u^5 + 5u^2 u_x^2 + u u_{xx}^2 + \frac{1}{14} u_{xxx}^2\right) dx. \end{aligned} \tag{40}$$

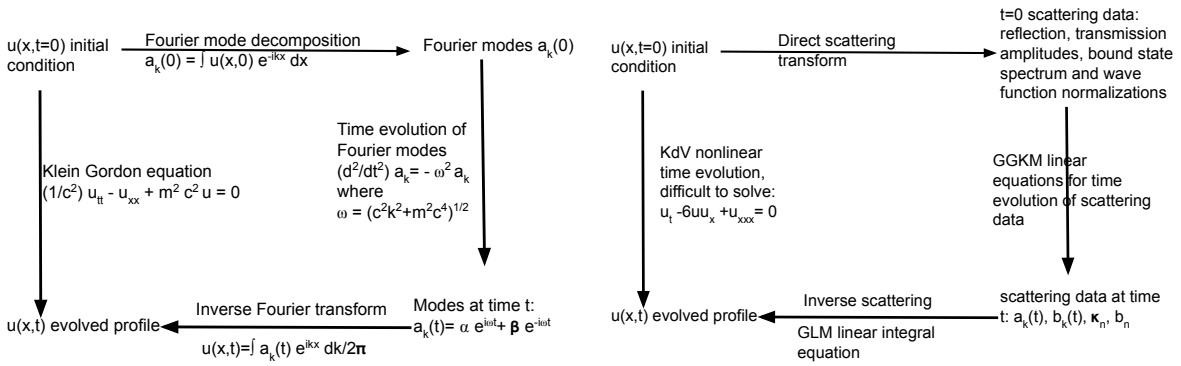

Figure 2: Schematic diagram of Fourier transform for solving Klein-Gordon initial value problem and inverse scattering transform for KdV initial value problem.

shock formation in 1d: the nonlinearity can steepen the u profile leading to shock-like discontinuities and multi-valued profiles. Adding a higher spatial derivative term can prevent singularity formation. The simplest such term is u_{xx}, but this leads to a dissipative equation $u_t + uu_x = \nu u_{xx}$ (if we ignore the nonlinear term, this is just the heat equation with an imaginary frequency-wavenumber dispersion relation). On the other hand, if we add a 3rd derivative term u_{xxx}, then we get non-dissipative but nonlinear dispersive evolution $u_t + uu_x = \alpha u_{xxx}$. Rescaling variables we arrive at the KdV equation in its standard form.

4.2 Similarity and Traveling wave solutions

Similarity solutions: It is easily verified that $u_t - 6uu_x + u_{xxx} = 0$ is invariant under the scale transformation

$$x \to \mu x, \quad t \to \mu^3 t \quad \text{and} \quad u \to \mu^{-2} u. \tag{41}$$

This may be used to seek scaling/similarity solutions of KdV. We notice that the combinations $\eta = x/(3t)^{1/3}$ and $ut^{2/3}$ are scale invariant. Thus, we seek solutions where u is of the form $u(x,t) = -(3t)^{-2/3} f(\eta)$ [The factors of 3 are for future convenience]. Substituting in (37) we obtain an ODE for f

$$f''' + (6f - \eta) f' - 2f = 0. \tag{42}$$

This 3rd order nonlinear ODE (with non-constant coefficients) may be reduced to a Painlevé equation (see § 7.2) and solved in terms of the Painlevé transcendents. It has a solution that decays as $\eta \to \infty$ and is oscillatory as $\eta \to -\infty$, generalizing the similarity solution of the linear dispersive equation $u_t + u_{3x} = 0$ that can be expressed in terms of the Airy function. There is believed to be a deeper connection between integrable PDEs and ODEs of Painlevé-type [14].

Traveling waves and solitary waves: Due to translation invariance in x, it is reasonable to expect KdV to admit traveling waves of permanent form $u(x,t) = f(\xi)$ where $\xi = x - ct$, f is the wave profile and c the velocity of propagation (rightward if $c > 0$). Indeed, the KdV equation was formulated to model Scott Russel's 'great wave of translation' [44]. Putting this traveling wave ansatz in (37) we get the ODE $-cf' - 6ff' + f''' = 0$ which may be readily integrated once in ξ to yield $f'' = 3f^2 + cf + A$, where A is an arbitrary integration constant. This equation admits a simple physical interpretation. If we regard ξ as time, it is Newton's 2nd law $mf'' = -V'(f)$ for a particle of mass $m = 1$ moving on the real

line with coordinate f. It is subject to the cubic potential $V(f) = -f^3 - \frac{c}{2}f^2 - Af$. Traveling wave profiles are then given by Newtonian trajectories in this potential. Bounded traveling waves correspond to bound trajectories of this fictitious particle, while unbounded profiles correspond to scattering states. It is instructive to plot V and find the qualitative nature of possible trajectories depending on the values of c and A. As usual, we may integrate Newton's equation once by multiplying by the integrating factor f':

$$\frac{1}{2}f'^2 - f^3 - \frac{c}{2}f^2 - Af = E \quad \Rightarrow \quad \xi - \xi_0 = \int_{f_0}^{f} \frac{df}{\sqrt{f^3 - \frac{c}{2}f^2 + Af + E}}. \tag{43}$$

The constant of integration E represents the energy of the particle. The determination of traveling wave profiles has thus been reduced to the inversion of an elliptic integral and allows us to write f as an elliptic function of ξ.

Bounded traveling waves are generically periodic, corresponding to the particle oscillating between two turning points. These are the so-called cnoidal waves, which exist over a whole range of energies E. However, when E is equal to a local maximum of the potential, the period of the wave diverges as the particles takes infinitely long to reach the point where $V' = 0$. The resulting wave is a solitary wave. To illustrate the simplest such possibility, we seek traveling waves that decay $f, f', f'' \to 0$ as $\xi = \pm\infty$, so that A and E are both zero. The equation reduces to $f'^2 = f^2(2f + c)$. For real solutions, we need $2f + c > 0$. What is more, taking $\xi \to \infty$, we infer that $c > 0$: waves that decay must be right moving. The resulting quadrature

$$\int_{f_0}^{f} \frac{d\tilde{f}}{\tilde{f}(2\tilde{f} + c)^{\frac{1}{2}}} = \pm \int_0^{\xi} d\tilde{\xi} \tag{44}$$

is easily performed using the substitution $f = -\frac{1}{2}c\,\mathrm{sech}^2\,\theta$, yielding $\mp\xi = \frac{2}{\sqrt{c}}(\theta - \theta_0)$, where $f_0 = -\frac{1}{2}c\,\mathrm{sech}^2\theta_0$. Thus, bounded decaying KdV traveling waves must be solitary waves of depression

$$u(x,t) = f(x - ct) = -\frac{1}{2}c\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c}(x - ct - x_0)\right). \tag{45}$$

The arbitrary constant x_0 is the position of the trough of the wave at $t = 0$. The wave of depression might have been anticipated on account of the negative sign in the advection term in $u_t - 6uu_x + u_{3x} = 0$. Reversing the sign of u in KdV leads to a solitary wave of elevation, which can model the great wave of translation seen by Scott Russel.

4.3 Hamiltonian and Lagrangian formulations

4.3.1 Gardner-Faddeev-Zakharov Poisson brackets

The KdV equation (say, on the real line with decaying BCs) admits a Hamiltonian formulation with

$$H = \int_{-\infty}^{\infty} \mathcal{H}\, dy = \int_{-\infty}^{\infty} \left[\frac{1}{2}u_y^2 + u^3\right] dy \tag{46}$$

and Gardner's Poisson brackets [23]

$$\{u(x), u(y)\} = \partial_x\delta(x - y) = -\partial_y\delta(x - y) = \frac{1}{2}(\partial_x - \partial_y)\delta(x - y). \tag{47}$$

In other words, Hamilton's equations $\dot{f} = \{f, H\}$ give the time evolution of any dynamical variable (functional f of u). In particular, one checks that

$$\begin{aligned}\{u(x), H\} &= \int [u_y\{u(x), u_y(y)\} + 3u^2(y)\{u(x), u(y)\}]dy = \partial_x \int (u_y\partial_y + 3u^2)\delta(x-y)dy \\ &= (-u_{xx} + 3u^2)_x = -u_{3x} + 6uu_x, \end{aligned} \tag{48}$$

so that $u_t = \{u, H\}$ is equivalent to (37). The PB can also be expressed between functionals of u:

$$\{F[u], G[u]\} = \int \frac{\delta F}{\delta u(x)} \frac{\delta G}{\delta u(y)} \{u(x), u(y)\}\, dxdy = \int \frac{\delta F}{\delta u(x)} \partial_x \left(\frac{\delta G}{\delta u(x)} \right) dx. \tag{49}$$

For instance, taking $G = H$, we get $\delta H/\delta u = -u_{xx} + 3u^2$ so that the evolution of any functional is:

$$\partial_t F = \int \frac{\delta F}{\delta u}(3u^2 - u_{xx})_x \, dx \quad \text{and in particular} \quad u_t = \partial_x \frac{\delta H}{\delta u(x)}. \tag{50}$$

We notice that Gardner's PB is not canonical. It is possible to introduce canonical variables by introducing the velocity potential $\phi(x,t)$ defined up to an additive constant[13] via $u = \phi_x$. Then, canonical PBs between ϕ and u imply Gardner's PB (47):

$$\{\phi(x), u(y)\} = \delta(x-y). \tag{51}$$

It turns out that there is another Hamiltonian formulation for the KdV equation based on Magri's PB. We will introduce it in §4.9

4.3.2 Lagrangian and Noether symmetries

The velocity potential and the canonical PBs (51) allow us to give a Lagrangian formulation for KdV. Indeed, it may be verified that KdV is the Euler-Lagrange equation for the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\phi_x\phi_t - \mathcal{H} = \frac{1}{2}\phi_x\phi_t - \phi_x^3 - \frac{1}{2}\phi_{xx}^2. \tag{52}$$

Indeed,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_t \frac{\partial \mathcal{L}}{\partial \phi_t} + \partial_x \frac{\partial \mathcal{L}}{\partial \phi_x} - \partial_{xx} \frac{\partial \mathcal{L}}{\partial \phi_{xx}} \quad \Rightarrow \quad \phi_{xt} - 6\phi_x\phi_{xx} + \phi_{xxxx} = 0 \tag{53}$$

is the KdV equation for $\phi_x = u$. We note that this Lagrangian is linear (rather than quadratic) in velocities. So the relation to the Hamiltonian is not via the usual Legendre transform. A more complete treatment uses Dirac brackets [47].

The above Lagrangian is invariant under constant translations of the field ϕ, position x and time t. Applying Noether's theorem, each of these leads to a (local) conservation law. The corresponding global conserved quantities are $M = \int u\, dx$, $P = \int u^2\, dx$ and $H = \int (u^3 + u_x^2/2)\, dx$ which we will interpret as the mass (or mean height of the water surface), momentum and energy.

[13] If we use decaying BCs for u ($u \to 0$ as $|x| \to \infty$), then ϕ must asymptotically approach (possibly different and nonzero) constant values as $|x| \to \infty$.

4.4 Lax pair (L, A) for KdV

In 1968, Peter Lax [31] proposed an (L, A) pair for the KdV equation. It consists of the linear second and third order differential operators

$$L = -\partial_x^2 + u(x,t) \quad \text{and} \quad A = 4\partial_x^3 - 6u\partial_x - 3u_x + A_0(t) = 4\partial_x^3 - 3[u,\partial]_+ + A_0(t), \tag{54}$$

where $A_0(t)$ is an arbitrary time-dependent multiple of the identity. L is the Schrödinger operator: its eigenvalue problem $L\psi = \lambda\psi$ coincidentally is the time-independent Schrödinger equation of quantum mechanics for a particle moving in the potential u with λ the energy eigenvalue in units where $\hbar^2/2m = 1$[14]. Differentiating in time, it follows that $L_t = u_t$. A little algebra shows that $[L, A] = 6uu_x - u_{xxx}$. Thus, the Lax equation $L_t = [L, A]$ is equivalent to the KdV equation (37)! The Lax representation can help us understand why KdV admits infinitely many conserved quantities. Roughly, L may be viewed as an infinite matrix, all of whose eigenvalues are conserved. Expressing these conserved quantities in terms of u is not straightforward but can be done by studying the quantum mechanical scattering problem for a particle in the potential $u(x,t)$. We will turn to this shortly.

4.4.1 Direct search for Lax operator A

Assuming $L = -\partial_x^2 + u$, it is possible to systematically search for the simplest differential operator A such that $L_t = [L, A]$ is equivalent to the KdV equation. We will do this here. In the next section, we will indicate where the Schrödinger operator came from and also give an alternative approach to finding A.

Now, if $u(x)$ is real, then L and consequently the scalar operator $L_t = u_t$ are hermitian. Thus, $[L, A]$ must be a hermitian scalar operator (i.e., multiplication by a function of x). For this, A must be anti-hermitian. The simplest possibility is a first order operator $c\partial_x + d$ where c and d are functions of x[15]. Anti-symmetrizing, we get

$$A_1 = (c\partial + d - \partial^\dagger c - d) = (c\partial + \partial c) = [c,\partial]_+ = (c' + 2c\partial). \tag{55}$$

where primes denote x-derivatives. We notice that $d(x)$ cancels out and may be ignored. Thus, using $[u,\partial] = -u'$ we get

$$[L, A_1] = [-\partial^2 + u, c' + 2c\partial] = -c''' - 4c''\partial - 4c'\partial^2 - 2cu'. \tag{56}$$

For this to be a scalar operator we must have $c' = c'' = 0$ at all x, i.e., c must be a constant. The resulting evolution equation is $u_t + 2cu_x = 0$. This is the 1d linear (chiral) wave equation for unidirectional propagation[16]. We need higher order A to get the KdV equation.

- The next simplest possibility is the anti-symmetrization of a 2nd order operator

$$A_2 = e\partial^2 + f\partial g\partial - (e\partial^2 - f\partial g\partial)^\dagger = e\partial^2 + f\partial g\partial - \partial^2 e - \partial g\partial f = -(e' + gf')' - 2(e' + gf')\partial. \tag{57}$$

However, we see that this is no different from (55) if we put $c = -(e' + gf')$.

[14] In 1d quantum systems, bound state energy eigenvalues are nondegenerate (see p. 99 of [25]), so that the corresponding eigenspaces of L are one-dimensional, see Eq. (32)

[15] Here, we are suppressing the possibility of adding to A an arbitrary function of time $A_0(t)$, which would commute with L and not affect the resulting evolution equation.

[16] The general solution $u = f(x - 2ct)$ for any differentiable function f describes waves of fixed profile that move to the right with speed $2c$ if $c > 0$.

Thus, we are led to consider the anti-symmetrization of a 3rd order operator (without terms with two derivatives). To begin with, let us consider the simplest possibility with b a *constant* below:

$$A_3 = b\partial^3 - (b\partial^3)^\dagger = b\partial^3 + \partial^3 b = 2b\partial^3 \tag{58}$$

We find, using

$$[L, A_3] = [-\partial^2 + u, 2b\partial^3] = -2b\left(u''' + 3u''\partial + 3u'\partial^2\right). \tag{59}$$

Thus, taking $A = A_3 + A_1 = 2b\partial^3 + 2c\partial + c'$, we get

$$[L, A] = -c''' - 2cu' - 2bu''' - (6bu'' + 4c'')\partial - (6bu' + 4c')\partial^2 \tag{60}$$

For this to be a scalar operator, we need $c' = -(3/2)bu'$ or $c = -(3/2)bu + c_0$ where c_0 is a constant. Thus, the Lax equation becomes

$$L_t = u_t = [L, A] = -\frac{b}{2}u''' + (3bu - 2c_0)u'. \tag{61}$$

Comparing with KdV $(u_t = 6uu_x - u_{3x})$, we infer that $b = 2$ and $c_0 = 0$ (non-zero c_0 corresponds to a Galilean boosted advecting velocity). In conclusion, we have discovered that $A = 4\partial^3 - 6u\partial - 3u' + A_0(t)$ along with the Schrödinger operator $L = -\partial^2 + u$ is a Lax pair for the KdV equation.

4.5 Miura transform, Riccati and mKdV equations

Here we will motivate and interpret the above KdV Lax pair. An important role in the study of the KdV equation was played by Miura's transformation between $u(x,t)$ and a new dependent variable $v(x,t)$,

$$u = v^2 + v_x. \tag{62}$$

Miura's transformation may be viewed as a Riccati equation (quadratically nonlinear first order ODE) for $v(x)$ with u appearing as a source/coefficient. The remarkable fact is that if v satisfies the modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{3x} = 0, \tag{63}$$

then u satisfies the KdV equation $u_t - 6uu_x + u_{3x} = 0$. mKdV differs from KdV only by the replacement of the advecting velocity by its square. Thus, the Miura transform is nearly a symmetry of the KdV equation (more on this when we discuss Bäcklund transformations in §7.1). In more detail, putting

$$\begin{aligned} u &= v^2 + v_x, \quad u_t = 2vv_t + v_{xt}, \quad u_x = 2vv_x + v_{xx}, \\ 6uu_x &= 6\left[2v^3 v_x + v^2 v_{xx} + 2vv_x^2 + v_x v_{xx}\right] \quad \text{and} \quad u_{3x} = 6v_x v_{xx} + 2vv_{3x} + v_{4x} \end{aligned} \tag{64}$$

in the KdV equation, we get

$$\begin{aligned} 2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2vv_{3x} + v_{4x} &= 0 \\ \text{or} \qquad \left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2 v_x + v_{3x}) &= 0. \end{aligned} \tag{65}$$

Thus, u solves KdV if v solves mKdV.

We may regard the above as a motivation for why one considers the Miura transformation in the context of KdV. Now, quite independent of the KdV equation, the Miura relation $v_x + v^2 = u$ can be re-written as the Schrödinger eigenvalue problem by introducing a new variable ψ in place of v. Indeed,

it is well-known that the Riccati equation $v_x + v^2 = u$ can be 'linearized' by the Cole-Hopf substitution $v = \psi_x/\psi$ [or conversely, $\psi(x) = \exp\int^x v(x')dx'$]. In fact, one readily obtains $v_x = \psi_{xx}/\psi - \psi_x^2/\psi^2$ so that $v^2 + v_x = \psi_{xx}/\psi$ so that $-\psi_{xx} + u\psi = 0$. The latter says that ψ lies in the kernel of the Schrödinger operator for a particle in the potential u. More generally, we get the Schrödinger eigenvalue problem $-\psi_{xx} + u\psi = \lambda\psi$ with non-zero eigenvalue λ if we shift $u \to u - \lambda$. More precisely, we perform a Galilean boost $x \mapsto x' = x + 6\lambda t$ and $u \mapsto u - \lambda$ which leaves the KdV equation invariant and consider the boosted Miura transform $u = v_{x'} + v^2 + \lambda$. The Cole-Hopf substitution $v = \psi_x/\psi$ then leads to the Schrödinger eigenvalue problem $-\psi_{xx} + u\psi = \lambda\psi$. Thus, we see the emergence of the Schrödinger operator in connection with the KdV equation.

Alternate approach to A: To see the emergence of the 3rd order differential operator A (54), we use the mKdV equation (63) to calculate the evolution of ψ. Using the Cole-Hopf transformation $v = \psi_x/\psi$ we find

$$\begin{aligned} v_t &= \frac{\psi_{xt}}{\psi} - \frac{\psi_x\psi_t}{\psi^2}, \quad v_x = \frac{\psi_{xx}}{\psi} - \left(\frac{\psi_x}{\psi}\right)^2 \\ \text{and} \quad v_{xxx} &= \frac{\psi_{xxxx}}{\psi} - 4\frac{\psi_{xxx}\psi_x}{\psi^2} - 3\left(\frac{\psi_{xx}}{\psi}\right)^2 + 12\frac{\psi_{xx}\psi_x^2}{\psi^3} - 6\left(\frac{\psi_x}{\psi}\right)^4. \end{aligned} \tag{66}$$

Inserting in (63) and reintroducing $u = \psi_{xx}/\psi$ we get

$$\begin{aligned} \partial_x\left(\frac{\psi_t}{\psi}\right) &= -6\frac{\psi_x^2\psi_{xx}}{\psi^3} - \frac{\psi_{xxxx}}{\psi} + 4\frac{\psi_{xxx}\psi_x}{\psi^2} + 3\left(\frac{\psi_{xx}}{\psi}\right)^2 = \partial_x\left(-4\frac{\psi_{xxx}}{\psi} + 3\frac{\psi_{xx}\psi_x}{\psi^2} + 3\frac{\psi_{xxx}}{\psi}\right) \\ &= \partial_x\left(\frac{-4\psi_{xxx} + 3u\psi_x + 3(u\psi)_x}{\psi}\right). \end{aligned} \tag{67}$$

It follows that

$$\psi_t = -4\psi_{xxx} + 3u\psi_x + 3(u\psi)_x + A_0(t)\psi = \left(-4\partial^3 + 3[u, \partial_x]_+ + A_0(t)\right)\psi. \tag{68}$$

We recognize the operator on the right as A from the Lax pair (54).

Remark: The above Miura transform $u = v^2 + v_x$ leads to the zero eigenvalue Schrodinger equation under the Cole-Hopf substitution $v = \psi_x/\psi$. To get the Schrodinger equation with eigenvalue λ we consider a Galilean boost taking $u \to u - \lambda$ and $x \to x' = x + 6\lambda t$ which leaves the KdV equation invariant. Thus we consider a 'boosted' Miura transform $u = v_{x'} + v^2 + \lambda$ where $x' = x + 6\lambda t$. Since KdV is invariant under Galilean boosts, we get the same mKdV equation for v: $v_t - 6v^2v_x + v_{xxx} = 0$. As before, we introduce ψ via the Cole-Hopf substitution ($v = \psi_{x'}/\psi$). To find A, we need to compute ψ_t/ψ from the mKdV equation. In this case we get

$$\partial_x\left(\frac{\psi_t - 6\lambda\psi_x}{\psi}\right) = \partial_x\left(\frac{-4\psi_{xxx} + 3[u, \partial_x]_+ - 6\lambda\psi_x}{\psi}\right). \tag{69}$$

The λ-dependent term on the RHS comes from replacing ψ_{xx}/ψ by $u - \lambda$ using the Schrödinger eigenvalue equation. On the LHS, the $-6\lambda\psi_x$ term comes from computing $\partial_t\psi'(x', t) = \psi_t + 6\lambda\psi_x$ for the Galilean boosted $x' = x + 6\lambda t$. The λ-dependent terms cancel and we get the same operator A as above.

4.6 Scattering data and its GGKM evolution

The first step in the IST approach to solving the KdV initial value problem is the direct scattering problem associated to the Schrödinger eigenvalue problem $L\psi = \lambda\psi$ for $L = -\partial_x^2 + u$. In general L can have

both a discrete and continuous spectrum of bound and scattering states. We will assume that $|u| \to 0$ sufficiently fast as $|x| \to \infty$ so that we may discuss quantum mechanical scattering in the potential u. We also assume that the potential supports only a finite number of bound states[17]. Though u may evolve in a complicated nonlinear way via the KdV equation, it turns out that certain 'scattering data' (essentially scattering amplitudes and bound state energy levels and wavefunction normalizations) in the potential $u(x,t)$ evolve simply (linearly) via the Gardner-Greene-Kruskal-Miura (GGKM) equations, which we now derive using the Lax pair L and $A = 4\partial^3 - 3[u,\partial]_+ + A_0(t)$ of (54).

4.6.1 Continuous spectrum

Consider scattering states of the Schrödinger operator with eigenvalue $\lambda = k^2$: $-\phi_{xx} + u(x,t)\phi = k^2\phi$. Since $u(x,0) \to 0$ as $|x| \to \infty$ sufficiently fast, λ can take all positive values at $t = 0$. Since $\int u^2 dx$ is independent of time, u must satisfy decaying BCs at all times and the continuous spectrum remains $\lambda \geq 0$ at all times. We are thus free to follow the evolution of the eigenfunction ϕ for any fixed k (so in effect, $\lambda_t = 0$). We will choose to study the scattering problem where (for $k > 0$), a plane wave 'enters'[18] from the right $(x = \infty)$ and choose an overall multiplicative constant so that ϕ has the 'Jost' asymptotic behavior (see Fig. 3):

$$\phi_k(x,t) \to \begin{cases} e^{-ikx} & \text{as} \quad x \to -\infty \\ a(k,t)e^{-ikx} + b(k,t)e^{ikx} & \text{as} \quad x \to \infty. \end{cases} \tag{70}$$

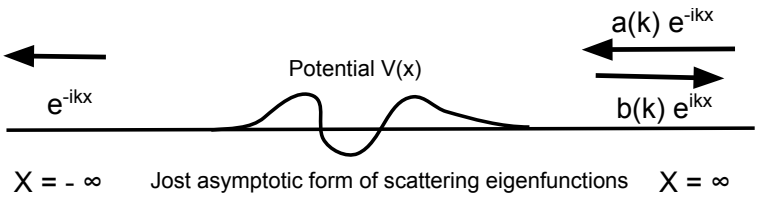

Figure 3: Jost asymptotic form for eigenfunctions of the Schrödinger operator in 1d potential scattering.

Thus, the reflection and transmission amplitudes are given by

$$r(k,t) = \frac{b(k,t)}{a(k,t)} \quad \text{and} \quad t(k,t) = \frac{1}{a(k,t)}. \tag{71}$$

We wish to find evolution equations for a and b following from KdV. Recall from (3.2) that if ϕ_k is a eigenfunction of L with eigenvalue $\lambda = k^2$, then by the time-independence of λ, $\dot{\phi} + A\phi$ is also an eigenfunction of L with the same eigenvalue. For KdV, $A = 4\partial^3 - 3[u,\partial]_+ + A_0(t)$ with $u \to 0$ as $|x| \to \infty$. From (70) we have

$$\dot{\phi}_k + A\phi_k \to (4ik^3 + A_0)e^{-ikx} = (4ik^3 + A_0)\phi_k \quad \text{as} \quad x \to -\infty. \tag{72}$$

Thus, $\dot{\phi}_k + A\phi$ and $(4ik^3 + A_0)\phi_k$ are both eigenfunctions of the Sturm-Liouville operator L with the same eigenvalue and asymptotic behavior. It follows that they must be equal

$$\dot{\phi}_k + A\phi_k = (4ik^3 + A_0)\phi_k \quad \text{for all} \;\; x \;\; \text{and} \;\; t. \tag{73}$$

[17] This is guaranteed if u satisfies the Faddeev condition $\int_{-\infty}^{\infty} |u(x)|(1+|x|)dx < \infty$.

[18] Here we use the terminology from quantum mechanics even though the time evolution is not necessarily of the $e^{-i\omega t}$ form. Time evolution is governed by KdV and not the time-dependent Schrodinger equation.

To find $\dot{a}$ and $\dot{b}$, we take the limit as $x \to \infty$ using (70) to get

$$\dot{a}e^{-ikx} + \dot{b}e^{ikx} + (4(-ik)^3 + A_0)ae^{-ikx} + (4(ik)^3 + A_0)be^{ikx} = (4ik^3 + A_0)(ae^{-ikx} + be^{ikx}) \text{ as } x \to \infty. \quad (74)$$

Comparing coefficients, we obtain the GGKM equations for the time-evolution of the scattering data for the continuous spectrum:

$$\dot{a}(k) = 0 \quad \text{and} \quad \dot{b}(k) = 8ik^3 b(k). \quad (75)$$

4.6.2 Discrete spectrum

The bound state wavefunctions $\phi_n(x)$ must satisfy $L\phi_n(x) = -\kappa_n^2 \phi_n(x)$ where we put $\lambda = -\kappa_n^2 < 0$. They are normalized to possess the asymptotic form

$$\phi_n(x) \to \begin{cases} e^{\kappa_n x} & \text{for } x \to -\infty \\ b_n e^{-\kappa_n x} & \text{for } x \to \infty. \end{cases} \quad (76)$$

This asymptotic behavior may be viewed as an analytic continuation of the Jost scattering states (70) to the upper-half of the complex k plane. More precisely, (70) reduces to the above asymptotic form if $k = i\kappa_n > 0$ and $a(i\kappa_n) = 0$ so that ϕ_n does not blowup as $x \to \infty$. In other words, the bound states occur at the zeros of $a(k)$ along the positive imaginary axis, which correspond to poles of the reflection and transmission amplitudes ($r = b/a$ and $t = 1/a$) in the complex wave number plane. We also note that ϕ_n are not normalized to unity but rather scaled to ensure the asymptotic behavior $\phi_n \to e^{\kappa_n x}$ as $x \to -\infty$. In fact, it can be shown that

$$\int_{\mathbb{R}} \phi_n^2 \, dx = ia'(i\kappa_n) b_n. \quad (77)$$

As a consequence, $a'(i\kappa_n) \neq 0$ so that $a(k)$ must have a simple zero at $k = i\kappa_n$.

Due to the isospectrality of L, κ_n are independent of time. Exercise: derive the evolution equation $\dot{b}_n = 8\kappa_n^3 b_n$ for b_n by comparing the asymptotic behaviors as $x \to \pm\infty$.

Recall from §3.2 that if ϕ_n is an eigenfunction of L with eigenvalue λ, then $\dot{\phi}_n + A\phi_n$ where $A = 4\partial^3 - 3[u, \partial]_+ + A_0(t)$ (54) is also an eigenfunction of L with the same eigenvalue. Now, we note that since $u \to 0$ as $x \to -\infty$,

$$\dot{\phi}_n \to 0 \quad \text{and} \quad A\phi_n \to (4\kappa_n^3 + A_0)e^{\kappa_n x} = (4\kappa_n^3 + A_0)\phi_n \quad \text{as} \quad x \to -\infty. \quad (78)$$

Thus,

$$\dot{\phi}_n + A\phi_n \to (4\kappa_n^3 + A_0(t))\phi_n \quad \text{as} \quad x \to \infty. \quad (79)$$

Consequently, $\dot{\phi}_n + A\phi_n$ and $(4\kappa_n^3 + A_0)\phi_n$ are both eigenfunctions of the Schrödinger operator L with the same eigenvalue and asymptotic behavior. Thus they must be equal for all x:

$$\dot{\phi}_n + A\phi_n = (4\kappa_n^3 + A_0)\phi_n \quad \forall \; x, t. \quad (80)$$

On the other hand,

$$\dot{\phi}_n \to \dot{b}_n e^{-\kappa_n x} \quad \text{and} \quad A\phi_n \to (-4b_n\kappa_n^3 + A_0 b_n)e^{-\kappa_n x} \quad \text{as} \quad x \to \infty. \quad (81)$$

Putting this in (80), we get as $x \to \infty$,

$$\dot{b}_n - 4b_n\kappa_n^3 + A_0 b_n)e^{-\kappa_n x} = (4\kappa_n^3 + A_0)b_n e^{-\kappa_n x}. \quad (82)$$

Comparing coefficients, we get an evolution equation for the normalization constant $\dot{b}_n = 8\kappa_n^3 b_n$, irrespective of the choice of $A_0(t)$. We note that this equation may also be obtained by analytic continuation of $\dot{b}(k) = 8ik^3 b(k)$ (75) for the continuous spectrum by putting $k = i\kappa_n$. Thus, the GGKM equations for the discrete spectrum are

$$\dot{\kappa}_n = 0 \quad \text{and} \quad \dot{b}_n = 8\kappa_n^3 b_n. \tag{83}$$

4.7 Inverse scattering: GLM equation

A key step in the IST approach to solving the KdV initial value problem is to return from the scattering data at time t to the potential or KdV wave height $u(x,t)$. Let us describe how this may be done. The scattering data (at time t) consists of the reflection amplitude $r(k) = b(k)/a(k)$, corresponding to the Jost asymptotic behavior

$$\phi(x) \to \begin{cases} e^{-ikx} & \text{as} \quad x \to -\infty \\ a(k)e^{-ikx} + b(k)e^{ikx} & \text{as} \quad x \to \infty \end{cases} \tag{84}$$

and the bound state spectrum $\lambda_n = -\kappa_n^2$ with $\kappa_n > 0$ for $n = 1, 2, \cdots, N$ corresponding to the asymptotic behavior

$$\phi_n(x) = \begin{cases} e^{\kappa_n x} & \text{as} \quad x \to -\infty \\ b_n e^{-\kappa_n x} & \text{as} \quad x \to \infty, \end{cases} \quad \text{with} \quad \int \phi_n(x)^2 dx = ia'(i\kappa_n)b_n. \tag{85}$$

Remarkably, $u(x,t)$ can be recovered from the scattering data. In fact, one may show that u is the derivative of a kernel $K(x,z)$ evaluated along its diagonal:

$$u(x) = -2\frac{d\hat{K}(x)}{dx} \quad \text{where} \quad \hat{K}(x) = K(x,x). \tag{86}$$

The kernel $K(x,z)$ is obtained by solving the Gelfand-Levitan-Marchenko (GLM) equation

$$K(x,z) + F(x+z) + \int_x^\infty K(x,y)F(y+z)dy = 0 \tag{87}$$

where the coefficient function $F(x)$ is determined by the scattering data:

$$F(x) = \sum_{n=1}^N \frac{b_n}{ia'(i\kappa_n)} e^{-\kappa_n x} + \int_{-\infty}^\infty r(k)e^{ikx}\frac{dk}{2\pi} \quad \text{where} \quad c_n^2 \equiv \frac{b_n}{ia'(i\kappa_n)}. \tag{88}$$

The continuous spectrum enters F through the Fourier transform of the reflection coefficient. F also includes a sum over the discrete spectrum of bound states corresponding to the imaginary wave numbers $k = i\kappa_n$.

• **Remark 1:** The quantity $c_n^2 = b_n/ia'(i\kappa_n)$ admits a simple interpretation. If the square of the norm of the bound state wavefunction $\tilde{\phi}_n$ is $\int |\tilde{\phi}_n(x)|^2 dx = 1$ (rather than $ia'(i\kappa_n)b_n$ for ϕ_n as above), then $\tilde{\phi}_n \to c_n e^{-\kappa_n x}$ as $x \to \infty$. In other words, $b_n = c_n\sqrt{ia'(i\kappa_n)b_n}$.

• **Remark 2:** It is noteworthy that the kernel $K(x,z)$ may also be used to obtain the scattering wavefunction [14]. For example, the scattering eigenfunction $\psi_k(x)$ of the Schrödinger operator with eigenvalue k^2 and asymptotic behavior e^{ikx} as $x \to \infty$ may be expressed as

$$\psi_k(x) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz \quad \text{with} \quad K(x,z), K_z(x,z) \to 0 \quad \text{as} \quad z \to \infty. \tag{89}$$

Thus, aside from the asymptotic behavior, $\psi_k(x)$ is a partial Fourier transform of $K(x,z)$.

We will now illustrate the use of the GLM equation in inverse scattering via two examples: scattering in a Dirac delta potential and solving the IVP for one and two soliton KdV initial states.

4.7.1 Direct and inverse scattering for a Dirac delta potential

To demonstrate how the GLM equation may be used to recover the potential from scattering data, we consider the example of an attractive Dirac delta potential $u(x) = -g\delta(x)$ with $g > 0$. The eigenfunctions $\psi(x)$ of $L = -\partial_x^2 - g\delta(x)$ are obtained by requiring that $\phi(0^+) = \phi(0^-)$ and that $\phi'(0^+) - \phi'(0^-) = -g\psi(0)$. There is precisely one bound state with wavefunction

$$\phi(x) = e^{-g|x|/2} \quad \text{with} \quad \lambda_1 = -\kappa_1^2 \quad \text{where} \quad \kappa_1 = \frac{g}{2}. \tag{90}$$

From the $x \to \infty$ behavior we read off $b_1 = 1$. Moreover, the corresponding unit norm wave function is $\sqrt{g/2}e^{-g|x|/2}$, so that the asymptotic behavior as $x \to \infty$ gives $c_1 = \sqrt{g/2}$. The scattering state wave functions with $\lambda = k^2$ are

$$\phi_k(x) = \begin{cases} t(k)e^{-ikx} & \text{for} \quad x < 0 \\ e^{-ikx} + r(k)e^{ikx} & \text{for} \quad x > 0, \end{cases} \tag{91}$$

with reflection and transmission amplitudes satisfying $1 + r = t$ and given by

$$r(k) = -\frac{g}{g + 2ik} \quad \text{and} \quad t(k) = \frac{2ik}{g + 2ik}, \quad \text{so that} \quad a = \frac{1}{t} = 1 + \frac{g}{2ik} \quad \text{and} \quad b = \frac{r}{t} = \frac{ig}{2k}. \tag{92}$$

This completes our discussion of the direct scattering problem. We note that r, t and a have simple poles at $k = i\kappa_1 = ig/2$.

We will now try to recover the Dirac delta potential by solving the GLM equation based on the above scattering data. In fact, from (88) we have

$$F(x) = \frac{g}{2}e^{-\frac{gx}{2}} - \frac{g}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{g + 2ik}dk = \frac{g}{2}e^{-\frac{gx}{2}} - \frac{g}{2\pi}\pi e^{-\frac{gx}{2}}\Theta(x > 0) = \frac{g}{2}e^{-\frac{gx}{2}}\Theta(x < 0). \tag{93}$$

The integral is performed via contour integration by exploiting the pole at $k = ig/2$ in the upper half plane.

The GLM equation (87) for the kernel K is then

$$K(x,z) + \frac{g}{2}e^{-\frac{g}{2}(x+z)}\Theta(x + z < 0) + \int_x^{\infty} K(x,y)\,\frac{g}{2}e^{\frac{g}{2}(y+z)}\Theta(y + z < 0)dy = 0. \tag{94}$$

For $x + z > 0$ both the Heaviside step functions vanish (since $y > x$), so $K(x,z) = 0$ for $x + z > 0$. For $x + z < 0$, the GLM equation becomes

$$K(x,z) + \frac{g}{2}e^{-\frac{g}{2}(x+z)} + \int_x^{-z} K(x,y)\frac{g}{2}e^{-\frac{g}{2}(y+z)}dy = 0. \tag{95}$$

Now it may be verified that the constant $K(x,z) = -g/2$ is a solution to this equation; it must be the only solution on account of uniqueness (see [14]). Thus, the GLM kernel is $K(x,z) = -\frac{g}{2}\Theta(x+z < 0)$.

It follows that $\hat{K}(x) = -\frac{g}{2}\Theta(x<0)$ and so from (86), $u(x) = g\Theta'(x<0) = -g\delta(x)$, recovering the attractive Dirac delta potential. We may also recover the scattering wavefunction from (89). For $x>0$, the integral does not contribute and so $\psi_k(x) = e^{ikx}$. On the other hand, for $x<0$, $K=-g/2$ for $z<-x>0$, so that

$$\psi_k(x) = e^{ikx} - \frac{g}{2}\int_x^{-x} e^{ikz}dz = e^{ikx}\left(1+\frac{g}{2ik}\right) + \frac{ig}{2k}e^{-ikx} \quad \text{for} \quad x<0. \tag{96}$$

We have thus recovered the scattering eigenfunction for a plane wave of unit amplitude moving to the right as $x\to\infty$. We may also read off the reflection and transmission amplitudes

$$r(k) = \frac{ig}{2k}\left(1+\frac{g}{2ik}\right)^{-1} \quad \text{and} \quad t(k) = \left(1+\frac{g}{2ik}\right)^{-1}, \tag{97}$$

recovering the direct scattering results of (92).

4.7.2 Reflectionless scattering: evolution of solitons

The function $F(x)$ (88) that enters the GLM equation (87) generally receives contributions from both the discrete and continuous spectrum for scattering in the potential $u(x)$. There are some special 'reflectionless' potentials for which $r(k)$ is zero for all k, so that the GLM equation simplifies. This is the case for certain (multi-)soliton initial states for the KdV equation. It turns out that the potential $u(x,0) = -N(N+1)\,\mathrm{sech}^2(x)$ for $N=1,2,3,\ldots$ displays reflectionless scattering (this is not true for general real values of N). These profiles, when evolved forward in time describe the evolution and scattering of N solitons. To illustrate the IST method of solving the KdV IVP, we will sketch the solution of the direct scattering problem for $N=1$ and $N=2$, use the GGKM equations to evolve the scattering data forward in time and then determine $u(x,t)$ by solving the GLM equation. We will discover that KdV solitary waves display phase-shift scattering.

One-soliton solution: Consider the initial wave of depression $u(x,0) = -2\,\mathrm{sech}^2 x$. In §4.2, we met this as the initial $(t=0)$ profile of a traveling solitary wave solution of the KdV equation with $c=4$. Here, we will use the inverse scattering transform to verify that $u(x,0)$ evolves into a solitary wave of permanent form. The Schrödinger eigenvalue problem

$$(-\partial_x^2 - 2\,\mathrm{sech}^2 x)\phi(x) = \lambda\phi(x) \tag{98}$$

may be transformed into the associated Legendre equation upon putting $T=\tanh x$:

$$((1-T^2)\phi')' + \left(2+\frac{\lambda}{1-T^2}\right)\phi = 0. \tag{99}$$

For a bounded solution ϕ to exist, we must have $\lambda = -\kappa_1^2 = -1$ or $\kappa_1 = 1$ with $\phi_1(x) = \frac{1}{2}\,\mathrm{sech} x = -\frac{1}{2}P_1^1(\tanh x)$. The multiplicative factor is chosen so that $\phi_1(x)\to e^{\kappa_1 x}$ as $x\to-\infty$. Consequently, $\phi_1 \to e^{-\kappa_1 x}$ as $x\to\infty$ so that $b_1 = 1$. Furthermore, the squared norm is

$$\int_{\mathbb{R}} \phi_1^2 dx = \frac{1}{2} = ia'(i\kappa_1)b_1 \quad \Rightarrow \quad ia'(i\kappa_1) = \frac{1}{2} \quad \text{and} \quad c_1 = \sqrt{2}. \tag{100}$$

In other words, the unit norm bound state wavefunction is

$$\tilde{\phi}_1 = \frac{\sqrt{2}}{2}\,\mathrm{sech}\, x \to \sqrt{2}e^{-\kappa_1 x} \quad \Rightarrow \quad c_1 = \sqrt{2}. \tag{101}$$

One verifies that scattering in this potential is reflectionless ($r(k) \equiv 0$). Thus, the GGKM evolution [(75) and (83)] of scattering data gives

$$r(k,t) \equiv 0, \quad b_1(t) = b_1(0)e^{8\kappa_1^3 t} = e^{8t}. \tag{102}$$

Thus, only the bound state contributes to the function F (88) appearing in the GLM equation

$$F(x,t) = \frac{b_1(t)}{ia'(i\kappa_1)}e^{-\kappa_1 x} = c_1^2 e^{-\kappa_1 x} = 2e^{8t-x}. \tag{103}$$

The GLM equation (87) for $K(x,z;t)$ becomes

$$K(x,z) + 2e^{8t-(x+z)} + 2\int_x^\infty K(x,y)e^{8t-(y+z)}dy = 0. \tag{104}$$

The appearance of e^{-z} suggests that we look for a factorized (rank one) kernel of the form $K(x,z;t) = L(x;t)e^{-z}$ leading to

$$L(x;t) + 2e^{8t-x} + 2L(x;t)e^{8t}\int_x^\infty e^{-2y}dy \quad \text{or} \quad L(x;t) = \frac{-2e^{8t-x}}{1+e^{8t-2x}}. \tag{105}$$

Thus

$$K(x,z) = \frac{-2e^{8t-x-z}}{1+e^{8t-2x}} \quad \text{and} \quad u(x,t) = -2\partial_x K(x,x) = -\frac{8e^{2x-8t}}{(1+e^{2x-8t})^2} = -2\,\mathrm{sech}^2(x-4t). \tag{106}$$

We see that u describes a traveling solitary wave with speed $c = 4$, as expected.

Two-soliton solution: Next we consider the case $N = 2$ of the IC $u(x,0) = -N(N+1)\,\mathrm{sech}^2 x = -6\,\mathrm{sech}^2 x$. As before, the Schrödinger eigenvalue problem becomes the associated Legendre equation

$$((1-T^2)\phi')' + \left(6 + \frac{\lambda}{1-T^2}\right)\phi = 0. \tag{107}$$

upon putting $T = \tanh x$. There are precisely two bound states, with eigenvalues $\lambda_n = \kappa_n^2$ with $\kappa_1 = 1$ and $\kappa_2 = 2$. We normalize the them so that $\phi_n \to e^{\kappa_n x}$ as $x \to -\infty$:

$$\phi_1 = \frac{1}{2}\tanh x\,\mathrm{sech}\,x \quad \text{and} \quad \phi_2 = \frac{1}{4}\,\mathrm{sech}^2 x. \tag{108}$$

Thus we may read off $b_1 = b_2 = 1$. Moreover, we find

$$\int \phi_1^2\,dx = \frac{1}{6} \quad \text{and} \quad \int \phi_2^2\,dx = \frac{1}{12} \quad \text{so that} \quad ia'(i\kappa_1) = \frac{1}{6} \ \text{ and } \ ia'(i\kappa_2) = \frac{1}{12} \tag{109}$$

and $c_1(0) = \sqrt{6}$ and $c_2(0) = 2\sqrt{3}$. The GGKM equations (83) imply $b_1(t) = e^{8t}$ and $b_2(t) = 3e^{64t}$. As before, the scattering is reflectionless, so $F(x,t) = 6e^{8t-x} + 12e^{64t-2x}$. The GLM integral equation

$$K(x,z,t) + 6e^{8t-(x+z)} + 12e^{64t-2(x+z)} + \int_x^\infty K(x,y,t)\left[6e^{8t-(y+z)} + 12e^{64t-2(y+z)}\right]dy = 0 \tag{110}$$

may be solved by assuming a rank 2 ansatz

$$K(x,z;t) = L_1(x,t)e^{-z} + L_2(x,t)e^{-2z}. \tag{111}$$

Comparing coefficients of e^{-z} and e^{-2z} we get

$$\begin{aligned} L_1 + 6e^{8t-x} + 6e^{8t}\left(L_1\int_x^\infty e^{-2y}dy + L_2\int_x^\infty e^{-3y}dy\right) &= 0, \\ L_2 + 12e^{64t-2x} + 12e^{64t}\left(L_1\int_x^\infty e^{-3y}dy + L_2\int_x^\infty e^{-4y}dy\right) &= 0 \end{aligned} \tag{112}$$

These are a pair of inhomogeneous linear equations for L_1 and L_2:

$$L_1 + 6e^{8t-x} + 3L_1e^{8t-2x} + 2L_2e^{8t-3x} = 0 \quad \text{and} \quad L_2 + 12e^{64t-2x} + 4L_1e^{64t-3x} + 3L_2e^{64t-4x} = 0, \tag{113}$$

with solutions

$$L_1(x,t) = 6(e^{72t-5x} - e^{8t-x})/D \quad \text{and} \quad L_2(x,t) = -12(e^{64t-2x} + e^{72t-4x})/D, \tag{114}$$

where $D(x,t) = 1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}$. Thus, the $N = 2$ KdV wave profile at time t is

$$\begin{aligned} u(x,t) &= -2\partial_x K(x,x) = -2(L_1e^{-x} + L_2e^{-2x})_x = 12\{(e^{8t-2x} + e^{72t-6x} + 2e^{64t-4x})/D\}_x \\ &= -12\frac{[3 + 4\cosh(2x-8t) + \cosh(4x-64t)]}{\{3\cosh(x-28t) + \cosh(3x-36t)\}^2}. \end{aligned} \tag{115}$$

4.8 Infinitely many local conserved quantities

From the GGKM equations (75) and (83) for the evolution of scattering data, we have discovered that the reciprocal of the transmission amplitude $a(k)$ (and bound state energies $\lambda_n = -\kappa_n^2$, if there are any) are independent of time. Here, we use these to construct an infinite sequence of conserved quantities for the KdV evolution that are local in the original field $u(x)$. This was originally done via a clever use of the Gardner transform, which is a slight extension of the Miura transform with a free parameter (k below). We will motivate it via scattering theory and a semiclassical expansion.

To proceed, we recall the asymptotic behavior (70) of the Jost eigenfunctions of the Schrödinger operator $L = -\partial_x^2 + u$:

$$\psi(x,k,t) = \begin{cases} e^{-ikx} & \text{as} \quad x \to -\infty \\ a(k)e^{-ikx} + b(k)e^{ikx} & \text{as} \quad x \to \infty. \end{cases} \tag{116}$$

This asymptotic form holds both for real k (scattering states) as well as for imaginary $k = i\kappa_n$ (bound states) in the upper-half complex k plane. Thus, we will, when needed, suppose that k has a positive imaginary part. The resulting equations can be justified independently.

We now choose to express ψ in terms of another function ρ via a formula that is reminiscent of the Jeffreys-WKB/Eikonal ansatz for the wavefunction in QM:

$$\psi(x) = \exp\left[-ikx + \int_{-\infty}^x \rho(y,k,t)\,dy\right]. \tag{117}$$

This form incorporates the e^{-ikx} behavior of (116) as $x \to -\infty$. Moreover, for ψ to remain bounded as $x \to \infty$, we need $\rho(y,k) \to 0$ as $y \to \infty$. Now let us impose the other boundary condition

$$e^{ikx}\psi \to a(k) + b(k)e^{2ikx} \quad \text{as} \quad x \to \infty. \tag{118}$$

For $\Im k > 0$, $e^{ikx} \to 0$ so we must have

$$a(k) = \lim_{x\to\infty} \exp\left[\int_{-\infty}^{x} \rho(y,k,t)\,dy\right] = \exp\left[\int_{-\infty}^{\infty} \rho(y,k,t)\,dy\right]. \tag{119}$$

Now since $a(k)$ is conserved, $\int_{\mathbb{R}} \rho(x,k,t)dx$ is conserved for any real k. So $\rho(x,k,t)$ is a conserved density for any k. To express these conserved quantities in terms of u, we first express the Schrödinger eigenvalue problem as an ODE for ρ. Indeed

$$\psi_x = (-ik+\rho)\,\psi \quad \text{and} \quad \psi_{xx} = \Big[\rho_x + (-ik+\rho(x,k))^2\Big]\,\psi \quad \Rightarrow \quad \rho_x+\rho^2-2ik\rho = u(x,t). \tag{120}$$

This 1st order quadratically nonlinear ODE for ρ may be viewed as a Riccati-type equation for ρ. It is invariant under the scale transformations

$$u \to \mu u, \quad \rho \to \sqrt{\mu}\rho, \quad x \to \mu^{-1/2}x \quad \text{and} \quad k \to \sqrt{\mu}k. \tag{121}$$

so that we may assign the scaling dimensions $1, 1/2, -1/2$ and $1/2$ to u, ρ, x and k. Given $u(x,t)$, it defines a 1 parameter family of functions $\rho(x,t)$ labelled by k. This transformation from u to ρ is the Gardner transform (in a slightly different form, it was proposed by Gardner to obtain infinitely many conserved quantities for KdV), it generalizes the Miura transform $v_x + v^2 = u$.

Alternative viewpoint on ρ as a conserved density: Putting (120) in the KdV equation, we find that ρ must satisfy

$$\begin{array}{rcl} (\partial_x + 2\rho - 2ik)\left(\partial_t\rho - 6(\rho^2 - 2ik\rho)\rho_x + \rho_{3x}\right) &=& 0 \quad \text{or} \\ (\partial_x + 2\rho - 2ik)\left(\partial_t\rho + \partial_x(\rho_{xx} + 6ik\rho^2 - 2\rho^3)\right) &=& 0. \end{array} \tag{122}$$

Thus, we see that *if* ρ satisfies the evolution equation

$$\partial_t\rho + \partial_x(\rho_{xx} + 6ik\rho^2 - 2\rho^3) = 0, \tag{123}$$

then u would satisfy the KdV equation (though not conversely). This evolution equation is in conservation form, so it would imply that $\int_{\mathbb{R}} \rho(x,k,t)dx$ is conserved in time for any k. This agrees with what we deduced from the conservation of $a(k)$. Thus, if we change our viewpoint and simply *postulate* the 'Gardner evolution equation' (123) for ρ which is related to u via the 'Gardner transform' (120), then u would satisfy the KdV evolution equation. This way, the conservation of $\int_{\mathbb{R}} \rho\,dx$ would lead us to conservation laws for KdV. Note that for $k = 0$, (123) reduces to the mKdV equation (63) and that (122) reduces to (65).

We may now use (120) to derive a sequence of local integrals of motion for KdV. As in the JWKB-Eikonal approximation, we expand ρ in a 'semi-classical' asymptotic series for large wave number k [19]:

$$\rho(x,k,t) = \sum_{n=1}^{\infty} \frac{\rho_n(x,t)}{(2ik)^n} \tag{124}$$

Note that ρ_n has the scaling dimension $(n+1)/2$ and is real[20]. Substituting this in (120) and comparing coefficients we get

$$\text{at} \quad \mathcal{O}(k^0) : \rho_1 = -u, \quad \text{at} \quad \mathcal{O}(1/k) : \rho_2 = \partial_x\rho_1 \quad \text{and}$$

[19] Recall that the semi-classical approximation to wave mechanics is valid when the de Broglie wavelength is small compared to the length scale over which the potential varies significantly, so that the classical point particle approximation holds.

[20] To see that ρ_n is real, we note that both $\rho(x,k)$ and $\rho^*(x,-k)$ satisfy the same Gardner transform equation (120) for real k. What is more, they must both vanish as $x \to \infty$, so that $\psi(x,k)$ and $\psi^*(x,-k)$ (117) are bounded as $x \to \infty$. Thus, they must be the same. Equating the series expansions (124) for $\rho(x,k)$ and $\rho^*(x,-k)$ we deduce that $\rho_n(x,t) = \rho_n^*(x,t)$.

$$\text{at} \quad \mathcal{O}(1/k^n) : \rho_{n+1} = \partial_x \rho_n + \sum_{m=1}^{n-1} \rho_m \rho_{n-m}. \tag{125}$$

Using this recursion relation we find

$$\begin{gathered}
\rho_1 = u, \quad \rho_2 = -u_x, \quad \rho_3 = u^2 - u_{xx}, \quad \rho_4 = (2(u^2) - u_{xx})_x, \\
\rho_5 = -u_{4x} + 2(u^2)_{xx} + u_x^2 + 2uu_{xx} - 2u^3, \quad \rho_6 = -u_{5x} + 18u_x u_{2x} + 8uu_{3x} - 16u^2 u_x, \\
\rho_7 = 5u^4 + 19(u_{2x})^2 - u_{6x} + 28u_x u_{3x} + 10uu_{4x} - 50uu_x^2 - 30u^2 u_{2x} \quad \text{or} \\
\rho_7 = 5u^4 + 10uu_x^2 + (u_{2x})^2 + \left[-u_{5x} - 30u^2 u_x + 18u_x u_{xx} + 10uu_{3x}\right]_x \quad \text{etc.}
\end{gathered} \tag{126}$$

Notice that ρ_2 and ρ_4 are exact differentials (show that this is true of ρ_6 as well) so that $\int \rho_2 \, dx$ and $\int \rho_4 \, dx$ vanish for decaying or periodic BCs. In fact, the even ρ_{2n} may all be shown to be exact differentials. Indeed, let us split $\rho = \rho_R + i\rho_I$ into its real and imaginary parts. Substituting in the Gardner transform (120) we find that

$$(\rho_R)_x + i(\rho_I)_x - 2ik\rho_R + 2k\rho_I + \rho_R^2 - \rho_I^2 + 2i\rho_R\rho_I = u. \tag{127}$$

The imaginary part of this equation gives a linear algebraic equation for ρ_R

$$\rho_R(2\rho_I - 2k) = -\partial_x \rho_I \quad \Rightarrow \quad \rho_R = -\frac{1}{2}\partial_x \log(\rho_I - k). \tag{128}$$

We see that ρ_R is an exact differential and $\int \rho_R \, dx$ does not lead to any conserved quantities. In fact, recalling that ρ_n are real, we may write

$$\rho_R = \sum_1^\infty \frac{\rho_{2n}}{(2ik)^n}, \quad \text{so} \quad \int \rho_{2n} \, dx = 0. \tag{129}$$

The odd coefficients ρ_{2n+1} lead to nontrivial conserved quantities which we denote

$$H_n = \frac{(-1)^{n+1}}{2} \int \rho_{2n+1} dx, \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{130}$$

with scaling dimensions $n+5/2$. We will show in (4.11) that these conserved quantities are in involution. Ignoring exact differentials in the densities, the first four nontrivial conserved quantities are

$$H_0 = \int \frac{u}{2} \, dx, \quad H_1 = \int \frac{u^2}{2} \, dx, \quad H_2 = \int \left(\frac{u_x^2}{2} + u^3\right) dx \quad \text{and} \quad H_3 = \frac{1}{2}\int \left[5u^4 + 10uu_x^2 + u_{2x}^2\right] dx. \tag{131}$$

The first three of these may be interpreted as the mass (mean height), momentum and energy of the KdV field. H_3 is a truly new conserved quantity that does not follow from any Noether symmetry.

4.9 Bi-Hamiltonian structure: Gardner and Magri brackets

KdV admits a second Hamiltonian formulation based on the Hamiltonian

$$H_2 = \frac{1}{2}\int u^2 \, dx, \tag{132}$$

which we have interpreted as the conserved momentum. Unlike the Gardner-Faddeev-Zahkarov Hamiltonian (46), H_2 is positive definite. It implies the KdV equation (37) upon use of Magri's PB [32]

$$\{u(x), u(y)\}_2 = \left(-\partial_x^3 + 2[\partial_x, u]_+\right) \delta(x - y) \tag{133}$$

in $u_t = \{u, H_2\}_2$. Here $[\partial_x, u]_+ = 2u_x + 4u\partial_x$. The operator on the right is reminiscent of but not the same as the third order Lax operator A (54). The PB between functionals of u can be expressed as

$$\{F[u], G[u]\}_2 = \int \frac{\delta F}{\delta u(x)} \left(-\partial_x^3 + 2u_x + 4u\partial_x\right) \frac{\delta G}{\delta u(x)}\, dx. \tag{134}$$

It may be verified that the Magri bracket is antisymmetric by integration by parts. Checking the Jacobi identity takes a bit more effort.

4.9.1 Relation of Magri's bracket to Virasoro Algebra

The Virasoro algebra spanned by the generators L_n (for $n \in \mathbb{Z}$) is a central extension of the Witt algebra of vector fields on the circle

$$i\{L_m, L_n\} = (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m}. \tag{135}$$

If we define the Fourier series with coefficients

$$u(x) = -\frac{1}{4} + \frac{6\pi}{c}\sum_{-\infty}^{\infty} L_n e^{-inx}, \tag{136}$$

then one verifies that u satisfies the Magri bracket up to a multiplicative constant

$$\{u(x), u(y)\} = \frac{6\pi}{c}\left(-\partial^3 + 2[u, \partial_x]_+\right)\delta(x - x'). \tag{137}$$

This relation of Virasoro to the Magri bracket was used by Gervais (1985) to construct infinitely many commuting charges using the Virasoro generators.

• Magri's bracket can be regarded as the Poisson bracket on the dual (coadjoint orbit) of the Virasoro Lie algebra.

4.10 Bi-Hamiltonian structure and KdV hierarchy

We may use the conserved quantities $H_{0,1,2\ldots}$ to define a family of evolution equations called the KdV hierarchy of PDEs. Let us introduce a sequence of evolutionary 'time' variables $t_{-1}, t_0, t_1, t_2 \ldots$ and corresponding evolution equations using the Gardner PB:

$$\partial_{t_n} u(x) = \{u, H_{n+1}\}_1 = \partial_x \frac{\delta H_{n+1}}{\delta u(x)} \quad \text{for} \quad n = -1, 0, 1, 2, \ldots \tag{138}$$

The first few evolution equations in this hierarchy are

$$u_{t_{-1}} = 0, \quad u_{t_0} = u_x, \quad u_{t_1} = 6uu_x - u_{3x}, \quad u_{t_2} = u_{5x} - 10uu_{3x} - 20u_x u_{xx} + 30u^2 u_x, \ldots \tag{139}$$

There is no evolution with respect to t_{-1}, while for t_0 we get the first order linear wave equation. Evidently, the standard time variable of KdV is $t = t_1$.

All these (generally nonlinear) PDEs are bi-Hamiltonian in the sense that they may each be given a second Hamiltonian formulation using the Magri bracket. In fact, one can show that for $n = -1, 0, 1, 2, \ldots$,

$$\partial_{t_n} u(x) = \{u, H_{n+1}\}_1 = \{u, H_n\}_2 \quad \text{or} \quad \partial_x \frac{\delta H_{n+1}}{\delta u(x)} = \left(-\partial_x^3 + 2[\partial_x, u]_+\right)\frac{\delta H_n}{\delta u(x)}, \tag{140}$$

where we have defined $H_{-1} = 0$. For $n = -1$ both sides vanish. For $n = 0$ both sides are equal to u_x while for $n = 1$ both sides equal $6uu_x - u_{3x}$. One can check (140) for higher n and also try to prove it, say by induction (assume that the equality holds for $n = -1, 0, 1, \cdots, m$ and then show that it holds for $n = m + 1$).

4.11 Conserved quantities in involution

We may use the bi-Hamiltonian structure (140) to show that the conserved quantities H_n are in involution with respect to both PBs

$$\{H_p, H_q\}_1 = 0 \quad \text{and} \quad \{H_p, H_q\}_2 = 0 \quad \text{for all} \quad p, q = 0, 1, 2, \ldots. \tag{141}$$

Let us show the involutive property with respect to the Gardner bracket; a similar argument works for the Magri bracket. We will first establish the identity

$$\{H_p, H_q\}_1 = \{H_{p-1}, H_{q+1}\}_1. \tag{142}$$

To see this we use (140), to write

$$\begin{aligned}
\{H_p, H_q\}_1 &= -\int \partial_x \frac{\delta H_p}{\delta u(x)} \frac{\delta H_q}{\delta u(x)} = -\int \left(-\partial_x^3 + 4u\partial_x + 2u_x\right) \frac{\delta H_{p-1}}{\delta u(x)} \frac{\delta H_q}{\delta u(x)} \\
&= -\int \frac{\delta H_{p-1}}{\delta u(x)} \left(\partial_x^3 - 4\partial_x \cdot u + 2u_x\right) \frac{\delta H_q}{\delta u(x)} = \int \frac{\delta H_{p-1}}{\delta u(x)} \left(-\partial_x^3 + 2u_x + 4u\partial_x\right) \frac{\delta H_q}{\delta u(x)} \\
&= \int \frac{\delta H_{p-1}}{\delta u(x)} \partial_x \frac{\delta Hq + 1}{\delta u(x)} dx = \{H_{p-1}, H_{q+1}\}_1,
\end{aligned} \tag{143}$$

where we have integrated by parts and used (140) a second time.

We will now show that $\{H_p, H_q\}_1 = 0$. A similar argument works for $\{\cdot, \cdot\}_2$. Without loss of generality, we may assume $p > q$. If $p - q$ is even, then we apply (142) repeatedly $(p - q)/2$ times to get $\{H_p, H_q\}_1 = \{H_{p-(p-q)/2}, H_{q+(p-q)/2}\}_1 = 0$. On the other hand, if $p - q$ is odd then repeated application of (142) $p - q$ times leads us to $\{H_p, H_q\}_1 = \{H_{p-(p-q)}, H_{q+(p-q)}\}_1 = \{H_q, H_p\}_1 =$ so that it must vanish due to anti-symmetry of the PB. Either way, the conserved quantities $H_0, H_1, \cdots$ are in involution.

4.12 Action-angle variables

We have shown that KdV admits infinitely many conserved quantities H_n in involution. It is plausible and it can be shown that they are independent, though we do not attempt a proof here. Though this is an infinite dimensional system, we might expect by extrapolating the results of the Liouville-Arnold theorem that a canonical transformation to action angle variables can be found. This is indeed the case. The action angle variables can be constructed using the scattering data.

In essence, action and angle variables for KdV are given by

$$P(k) = \frac{2k}{\pi} \log |a(k)| \quad \text{and} \quad Q(k) = \arg b(k) \tag{144}$$

supplemented by additional action-angle variables from the discrete part of the spectrum.

5 Zero curvature representation

The zero curvature representation (the name will be justified shortly) generalizes the idea of a Lax pair to a wider class of nonlinear evolution equations (especially for systems in one spatial dimension). Let us introduce it in the context of the KdV equation.

5.1 Lax pair to zero curvature for KdV

To understand how the zero curvature representation arises, we change our viewpoint and regard the nonlinear Lax equation $L_t = [L, A]$ as the compatibility condition for the pair of linear equations

$$L\psi = \lambda\psi \quad \text{and} \quad \psi_t = -A\psi, \quad \text{where} \quad \lambda \quad \text{is a constant.} \tag{145}$$

Indeed, by differentiating $L\psi = \lambda\psi$ in time and using the second equation, it is verified that for the eigenvalue λ of L to be time-independent, L and A must satisfy the Lax equation $L_t = [L, A]$.

In the case of the KdV equation (37), $L = -\partial_x^2 + u$ involves 2nd order space derivatives, so that the two equations in (145) are somewhat unsymmetrical. There is a way of replacing (145) with a more symmetrical pair of linear equations involving only 1st order derivatives[21]:

$$\partial_x F = UF \quad \text{and} \quad \partial_t F = VF. \tag{146}$$

The price to be paid is that U and V are now square matrices and F a column vector (of size equal to the order of the differential operator L). Here, roughly, U and V play the role of a Lax pair. Their matrix elements depend on the dynamical variables (such as u for KdV) as well as on the eigenvalue λ, which is now called the spectral parameter.

Eqn. (146) is called the auxiliary linear system of equations. The vector space V in which F resides is called the auxiliary linear space. If we regard $\partial_x - U$ and $\partial_t - V$ as the space and time components of a covariant derivative, then the auxiliary linear equations require that every vector field $F(x,t)$ is covariantly constant. The auxiliary linear system is overdetermined in the sense that U and V must satisfy a compatibility condition for solutions F to exist: there would be no non-zero solutions otherwise. Indeed, equating mixed partials $\partial_x\partial_t F = \partial_t\partial_x F$, we get the consistency condition

$$\partial_t U - \partial_x V + [U, V] = 0. \tag{147}$$

The original nonlinear evolution equations are said to have a zero curvature representation if they are equivalent to (147). Before explaining how this scheme may be used to find conserved quantities, let us use the KdV equation to provide an example.

To find U for KdV, we write the eigenvalue problem of the Lax operator $(-\partial_x^2 + u)\psi = \lambda\psi$ as a pair of first order equations by introducing the column vector $F = (f_0, f_1)^T = (\psi, \psi_x)^T$:

$$\partial_x \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \quad \Rightarrow \quad U = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}. \tag{148}$$

Next, we use $\psi_t = -A\psi$ with $A\psi = 4\psi_{xxx} - 6u\psi_x - 3u_x\psi$ to find V such that $\partial_t F = VF$. We may express $\psi_t = -A\psi$ as a system of two first order ODEs. First, we use $L\psi = -\psi_{xx} + u\psi = \lambda\psi$ to

[21] The passage from the KdV Lax pair (L, A) to (U, V) is akin to that from Newton's second order equation to Hamilton's first order equations.

eliminate the third order derivative[22] appearing in $A\psi$, i.e.,

$$-\psi_{xxx} = -u_x\psi + (\lambda - u)\psi_x \quad \Rightarrow \quad A\psi = -2(u + 2\lambda)\psi_x + u_x\psi. \tag{149}$$

Next, using $F = (\psi, \psi_x)^T$, $\psi_t = -A\psi$ takes the form

$$\partial_t \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = V \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \text{ with } V = \begin{pmatrix} -u_x & 2(u+2\lambda) \\ 2u^2 - u_{xx} + 2u\lambda - 4\lambda^2 & u_x \end{pmatrix}. \tag{150}$$

Here, the second row of the matrix V is obtained by taking the x derivative of the first row and using $\psi_{xx} = (u - \lambda)\psi$.

The parameter λ that appears in U and V originally arose as the eigenvalue of the Lax operator L. This explains the name *spectral parameter*. More generally, a zero curvature representation need not arise from a Lax pair and the corresponding spectral parameter λ may not admit an interpretation as an eigenvalue.

5.2 Conserved quantities from the zero curvature condition

Here, we will learn how the zero curvature representation may be used to construct conserved quantities. Let us consider the first of the auxiliary linear equations in (146) for the column vector F: $\partial_x F = U(x)F(x)$. Let us imagine solving this equation for F from an initial location x to a final point y. If $y = x + \delta x$, then for small δx,

$$F(x + \delta x) \approx [\mathbf{1} + \delta x\, U(x)]\; F(x). \tag{151}$$

More generally, linearity suggests that the solution may be written as $F(y) = T(y, x)F(x)$. Here $T(y, x)$ may be viewed as transforming $F(x)$ into $F(y)$ and is called the *transition matrix*[23]. For this to work, $T(y, x)$ must satisfy the equation and 'boundary' condition

$$\frac{\partial}{\partial y} T(y, x; \lambda) = U(y; \lambda)T(y, x; \lambda) \quad \text{and} \quad T(x, x; \lambda) = \mathbf{1}. \tag{152}$$

This is obtained by inserting $F(y) = TF(x)$ in the auxiliary linear equation and requiring it to hold for any F. In §5.3, we learn that $T(y, x)$ may be expressed (essentially by iterating (151)) as an *ordered exponential series* which we abbreviate as

$$T(y, x; \lambda) = \mathrm{P} \exp \int_x^y U(z; \lambda)\, dz. \tag{153}$$

For simplicity, we henceforth suppose that our one-dimensional system is defined on the spatial interval $-a \le x \le a$ with periodic boundary conditions, so that $U(-a) = U(a)$ and $V(-a) = V(a)$. Thus, we may view our spatial coordinate as parametrizing a circle of circumference $2a$. So far, we have been working at one instant of time. It turns out that the transition matrix around the full circle ($x = -a$ to $y = a$), also called the monodromy matrix,

$$T_a(t, \lambda) = \mathrm{P} \exp \int_{-a}^{a} U(z; t, \lambda)\, dz \tag{154}$$

[22] For a Lax operator which is an n^{th}-order spatial differential operator, we may express the Lax equation as a system of n first order equations for the column vector $(\psi, \psi_x, \psi_{xx}, \cdots, \psi_{(n-1)x})^T$ comprising the first $(n-1)$ derivatives of the eigenfunction ψ. Here T denotes the transpose. U and V then become $n \times n$ matrices. For the KdV equation, $n = 2$.

[23] The transition matrix $T(y, x)$ transforms the vector F at x to its value at y. Such a matrix is sometimes called a parallel transport operator or a Wilson line. The parallel transport around a closed curve is called the holonomy of the connection or monodromy matrix. The trace of the holonomy or monodromy matrix is called the Wilson loop.

has remarkably simple time evolution[24]. In fact, using the ordered exponential series (153) and the zero curvature condition (147), one may show that the transition matrix evolves according to

$$\partial_t T(y,x;t) = V(y;t)T(y,x;t) - T(y,x;t)V(x;t). \tag{155}$$

Specializing to $x=-a$ and $y=a$, we obtain an evolution equation for the monodromy matrix $T_a(t) = T(a,-a;t)$:

$$\partial_t T_a(t,\lambda) = [V(a;t,\lambda), T_a(t,\lambda)]. \tag{156}$$

We are now in familiar territory: this equation has the same structure as the Lax equation (25) upon making the replacements $T_a \mapsto L$ and $V \mapsto -A$. As in §3.2, this immediately implies that the trace of the monodromy is independent of time. Moreover, this is true for any value of the spectral parameter λ. Thus, if we expand $\operatorname{tr} T_a(\lambda)$ in a series in (positive and negative) powers of λ, then each of the coefficients is a conserved quantity. In many interesting cases such as the KdV and nonlinear Schrödinger equations, one obtains infinitely many conserved quantities in this way.

Why the name 'zero curvature'?

Einstein's theory of gravity teaches us that a gravitational field is associated to space-time curvature. It turns out that an electromagnetic field is *also* associated to curvature, though not of space-time but of an 'internal' space. Now, the electric and magnetic fields may be packaged in the components of the field strength: $F_{0i} = E_i/c$ and $F_{ij} = \sum_k \epsilon_{ijk} B_k$ for $1 \le i,j,k \le 3$, where c denotes the speed of light. Thus, the field strength is a measure of curvature. What is more, specializing to one spatial dimension and introducing the scalar and vector potentials A_0 and A_1, we have $F_{01} = \partial_t A_1 - \partial_x A_0$. More generally, in the non-abelian version of electromagnetism relevant to the strong and weak interactions, A_0 and A_1 become matrices and the field strength acquires an extra commutator term: $F_{01} = \partial_t A_1 - \partial_x A_0 + [A_1, A_0]$. Now making the substitutions $A_1 \to U$ and $A_0 \to V$, we see that the consistency condition (147) states that the field strength or curvature of this generalized electromagnetic field vanishes. Hence the name zero curvature condition.

5.3 Path ordered exponential

Consider the linear system of ODEs for the transition matrix $T(y,x)$:

$$\partial_y T(y,x) = U(y)T(y,x) \quad \text{with} \quad T(x,x) = I. \tag{157}$$

For each x and y, T is a matrix and so is $U(y)$. The difficulty in solving this equation lies in the fact that U is not a constant matrix and worse, the matrices $U(y)$ may not commute at distinct values of y. If they did, then the solution is just an ordinary exponential: $T(y,x) = \exp \int_x^y U(z)dz$. However, this formula does not give the solution when $[U(z), U(z')] \neq 0$ for $x < z, z,' \le y$. (a) Convert this system of ODEs into an integral equation (what do you integrate with respect to and from where to where?). Show that you get

$$T(y,x) = I + \int_x^y U(z)T(z,x)\, dz. \tag{158}$$

[24]The time derivative of the transition matrix $T(y,x)$ is not quite a commutator. On the other hand, just like the Lax matrix L, the monodromy matrix $T(a,-a)$ evolves via a commutator when periodic boundary conditions are imposed.

(b) By repeated use ('Picard iteration') of this formula obtain a series representation for the transition matrix

$$T(y,x) = \sum_{n=0}^{\infty} \int \cdots \int_{x<z_n<\cdots<z_1<y} dz_1 \cdots dz_n \, U(z_1)U(z_2)\cdots U(z_n). \tag{159}$$

(c) Now, define path ordering denoted by the symbol P via

$$\mathrm{P}(U(z_1)U(z_2)) = \begin{cases} U(z_1)U(z_2) & \text{if} \quad z_1 \geq z_2 \\ U(z_2)U(z_1) & \text{if} \quad z_2 \geq z_1, \end{cases} \tag{160}$$

Essentially the earlier locations are placed to the right. Argue that

$$\int_{z_1>z_2} dz_1 dz_2 U(z_1)U(z_2) = \int_{z_2>z_1} dz_1 dz_2 U(z_2)U(z_1) \tag{161}$$

and thereby show that

$$\int_x^y dz_1 \int_x^{z_1} dz_2 U(z_1)U(z_2) = \frac{1}{2}\int_x^y dz_1 \int_x^y dz_2 \, \mathrm{P}(U(z_1)U(z_2)). \tag{162}$$

Essentially, the integral over a triangle has been expressed as half the integral over a square. Proceeding this way, one can write the above series in a manner reminiscent of the exponential series

$$\int_x^y dz_1 \int_x^{z_1} dz_2 \cdots \int_x^{z_{n-1}} dz_n \, U(z_1)U(z_2)\cdots U(z_n) = \frac{1}{n!}\int_x^y \cdots \int_x^y dz_1 dz_2 \cdots dz_n \mathrm{P}(U(z_1)U(z_2)\cdots U(z_n)) \tag{163}$$

so that

$$T(y,x) = \sum_0^{\infty} \frac{1}{n!}\int_x^y \cdots \int_x^y dz_1 dz_2 \cdots dz_n \mathrm{P}(U(z_1)U(z_2)\cdots U(z_n)) =: \mathrm{P}\exp\left[\int_x^y U(z)dz\right]. \tag{164}$$

This series is called the path-ordered exponential and denoted $\mathrm{P}\exp$. The last expression is just a short form and is defined by the series.

A similar series with space coordinates replaced with time is called the time-ordered exponential. Such a series appears in the various places in physics: the time evolution operator in quantum mechanical problems with a time-dependent Hamiltonian, parallel transport operators in geometry, the Dyson or Born series, etc.

6 Nonlinear Schrödinger field

Here we consider the one-dimensional nonlinear Schrödinger equation (NLSE) [25] for the complex scalar field $\psi(x,t)$:

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + 2\kappa|\psi|^2\psi. \tag{165}$$

Here, κ is a real parameter. For $\kappa = 0$, it reduces to the linear Schrödinger wave equation (in units where $\hbar = 1$) for a free quantum mechanical particle of mass $m = 1/2$ moving on a line. The NLSE is used to

[25] Like KdV, NLSE too admits solitary wave solutions. They are called bright and dark solitons depending on whether κ is negative or positive corresponding to attractive or repulsive interactions among the bosons.

model a gas of bosons with short-range (contact or delta function) pairwise interactions (of strength κ) in a 'mean field' approximation where $|\psi|^2$ is interpreted as the density of bosons at location x. These bosons could for instance be alkali metal atoms such Lithium or Rubidium, especially at low temperature. At low energies, the scattering between atoms is dominated by S-wave scattering which is the the only partial wave for which the amplitude for a particle to be at the origin is non-zero. The equation (possibly with an external trapping potential) is also called the Gross-Pitaevskii equation and may be considered in 1, 2, or 3 dimensions. When quantized, ψ and ψ^* become field annihilation and creation operators satisfying bosonic commutation relations. Contrast this with the linear Schrödinger equation of QM where ψ is a probability amplitude describing just one particle. NLSE also has applications in nonlinear optics.

NLSE is an infinite dimensional Hamiltonian system with Hamiltonian

$$H = \int \left(|\psi_x|^2 + \kappa|\psi|^4\right) dx \tag{166}$$

and canonical PBs

$$\{\psi(x), \psi^*(y)\} = -i\delta(x-y), \qquad \{\psi(x), \psi(y)\} = \{\psi^*(x), \psi^*(y)\} = 0. \tag{167}$$

6.1 Zero curvature representation for NLSE

The NLSE admits a zero curvature representation $U_t - V_x + [U, V] = 0$ (147) if the U and V matrices are chosen as [18]

$$\begin{aligned} U &= \sqrt{\kappa}(\psi^*\sigma_+ + \psi\sigma_-) + \lambda\frac{\sigma_3}{2i} \qquad \text{and} \\ V &= i\kappa|\psi|^2\sigma_3 - i\sqrt{\kappa}\,(\psi_x^*\sigma_+ - \psi_x\sigma_-) - \lambda\sqrt{\kappa}(\psi^*\sigma_+ + \psi\sigma_-) - \lambda^2\frac{\sigma_3}{2i}. \end{aligned} \tag{168}$$

Here $\sigma_\pm = (1/2)(\sigma_1 \pm i\sigma_2)$ are built from the Pauli matrices σ_1 and σ_2. Thus, they are linear operators on the auxiliary linear space $V = \mathbb{C}^2$ (the vector space V is not to be confused with the time component of the flat connection.). Recall that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{169}$$

6.2 Fundamental Poisson brackets and classical r-matrix

As discussed before, the trace of the monodromy matrix constructed from U allows us to find conserved quantities. To find conserved quantities in involution, we need to compute their PBs. The starting point for this is to have the PBs between matrix elements of U. We check that

$$\{U(x,\lambda) \overset{\otimes}{,} U(y,\mu)\} = i\kappa(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)\delta(x-y) \tag{170}$$

It is noteworthy that the RHS is independent of the spectral parameters λ, μ. Here, the matrix elements of a tensor product are defined as $(A \otimes B)_{ij;kl} = A_{ik}B_{jl}$. Remarkably, these PBs can be expressed as a commutator with the so-called classical r-matrix:

$$\{U(x,\lambda) \overset{\otimes}{,} U(y,\mu)\} = [r(\lambda-\mu), U(x,\lambda) \otimes I + I \otimes U(x,\mu)]\delta(x-y). \tag{171}$$

We will see that this commutator form facilitates finding conserved quantities in involution because the trace of a commutator vanishes. These 'fundamental PBs' are said to be ultralocal due to the absence of derivatives of Dirac delta functions on the RHS. The r-matrix is a linear self-map on $V \otimes V$, in a given basis it has 4 indices r_{ijkl}. The NLSE r-matrix is a scalar multiple of a the permutation matrix:

$$r(\lambda - \lambda') = \frac{\kappa P}{2(\lambda - \lambda')}. \tag{172}$$

Unlike for the harmonic oscillator, this r-matrix is non-dynamical: it is independent of ψ, ψ^*. The permutation matrix P is defined by $P(u \otimes v) = v \otimes u$. It is clear that $P^2 = P$ so that $P^{-1} = P$. Taking the ij matrix element on either side we may arrive at the matrix elements of P:

$$P_{ijkl} u_k v_l = v_i u_j \quad \text{for all} \quad u, v \quad \Rightarrow \quad P_{ijkl} = \delta_{il}\delta_{jk} \tag{173}$$

P acts on tensor products of operators on $\mathbb{C}^2$ by conjugation:

$$P(A \otimes B) = (B \otimes A)P \quad \text{so that} \quad [P, A \otimes B + B \otimes A] = 0. \tag{174}$$

Here, the matrix elements of a tensor product are defined as $(A \otimes B)_{ij;kl} = A_{ik}B_{jl}$.

The permutation operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be expressed in terms of Pauli matrices: $P = \frac{1}{2}(I + \sum_{a=1}^{3} \sigma_a \otimes \sigma_a)$. This is verified as follows:

$$P_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + (\sigma_1)_{ik}(\sigma_1)_{jl} + (\sigma_2)_{ik}(\sigma_2)_{jl} + (\sigma_3)_{ik}(\sigma_3)_{jl}). \tag{175}$$

Now, using the matrix elements of the Pauli matrices,

$$(\sigma_1)_{ik} = (1 - \delta_{ik}), \quad (\sigma_2)_{ik} = (i\delta_{i2}\delta_{2k} - i\delta_{i1}\delta_{1k}) \quad \text{and} \quad (\sigma_3)_{ik} = \delta_{ik} - 2\delta_{i2}\delta_{2k}, \tag{176}$$

we get

$$\begin{aligned} P_{ijkl} &= \frac{1}{2}(3\delta_{ik}\delta_{jl} + 1 - \delta_{ik} - \delta_{jl} - \delta_{i2}\delta_{1k}\delta_{j2}\delta_{1l} + \delta_{il}\delta_{2k}\delta_{j2}\delta_{1l} \\ &+ \delta_{i2}\delta_{1k}\delta_{j1}\delta_{2l} - \delta_{i1}\delta_{2k}\delta_{j1}\delta_{2l} - 2\delta_{i2}\delta_{2k}\delta_{jl} - 2\delta_{ik}\delta_{j2}\delta_{2l} + 4\delta_{i2}\delta_{2k}\delta_{j2}\delta_{2l}) = \delta_{il}\delta_{jk}. \end{aligned} \tag{177}$$

6.2.1 Anti-symmetry of the fundamental PBs

• If the r-matrix satisfies a certain skew-symmetry property, then the fundamental PBs are guaranteed to be anti-symmetric. This condition may be expressed in a variety of ways, using indices or the permutation operator or using a subscript notation:

$$r(\lambda - \lambda')_{ikpl} = -r(\lambda' - \lambda)_{kilp} \quad \text{or} \quad Pr(\lambda' - \lambda)P = -r(\lambda - \lambda') \quad \text{or} \quad Pr_{21}P = -r_{12}. \tag{178}$$

Let us consider each of these formulations.

Index Notation: In index notation, anti-symmetry of the fundamental PBs is the statement,

$$\{U_{ij}(x, \lambda), U_{kl}(y, \lambda')\} = -\{U_{kl}(y, \lambda'), U_{ij}(x, \lambda)\}. \tag{179}$$

To find a condition on r that would guarantee this, we write out the LHS and RHS:

$$\begin{aligned} \{U_{ij}(x, \lambda), U_{kl}(y, \lambda')\} &= \{U(x, \lambda) \otimes, U(y, \lambda')\}_{ikjl} = [r(\lambda - \lambda'), U(x, \lambda) \otimes I + I \otimes U(x, \lambda')]_{ikjl}\delta(x - y) \\ &= r(\lambda - \lambda')_{ikpq}(U(x, \lambda) \otimes I + I \otimes U(x, \lambda'))_{pqjl} - (U(x, \lambda) \otimes I + I \otimes U(x, \lambda'))_{ikpq}r(\lambda - \lambda')_{pqjl} \end{aligned}$$

$$
\begin{aligned}
& = r(\lambda-\lambda')_{ikpq}(U(x,\lambda)_{pj}\delta_{ql}+\delta_{pj}U(x,\lambda')_{ql})-(U(x,\lambda)_{ip}\delta_{kq}+\delta_{ip}U(x,\lambda')_{kq})r(\lambda-\lambda')_{pqjl}\\
& = r(\lambda-\lambda')_{ikpl}U(x,\lambda)_{pj}+r(\lambda-\lambda')_{ikjq}U(x,\lambda')_{ql}-U(x,\lambda)_{ip}r(\lambda-\lambda')_{pkjl}-U(x,\lambda')_{kq}r(\lambda-\lambda')_{iqjl}\\
-\{U_{kl}(y,\lambda'),U_{ij}(x,\lambda)\} & = -\{U(y,\lambda')\otimes,U(x,\lambda)\}_{kilj}=[-r(\lambda'-\lambda),U(y,\lambda')\otimes I+I\otimes U(y,\lambda)]_{kilj}\delta(x-y)\\
& = -r(\lambda'-\lambda)_{kipq}(U(x,\lambda')\otimes I+I\otimes U(x,\lambda))_{pqlj}-(U(x,\lambda')\otimes I+I\otimes U(x,\lambda))_{kipq}r(\lambda'-\lambda)_{pqlj}\\
& = -r(\lambda'-\lambda)_{kipq}(U(x,\lambda')_{pl}\delta_{qj}-\delta_{pl}U(x,\lambda)_{qj})+(U(x,\lambda')_{kp}\delta_{iq}+\delta_{kp}U(x,\lambda)_{iq})r(\lambda'-\lambda)_{pqlj}\\
& = -r(\lambda'-\lambda)_{kipj}U(x,\lambda')_{pl}-r(\lambda'-\lambda)_{kilq}U(x,\lambda)_{qj}+U(x,\lambda')_{kp}r(\lambda'-\lambda)_{pilj}+U(x,\lambda)_{iq}r(\lambda'-\lambda)_{kqlj}
\end{aligned}
$$

Comparing, we see that $r(\lambda-\lambda')_{ikpl}=-r(\lambda'-\lambda)_{kilp}$ would guarantee anti-symmetry of the PBs.

Permutation notation: The anti-symmetry of FPBs may be expressed using the permutation operator

$$\{U(x,\lambda)\overset{\otimes}{,}U(y,\lambda')\}=-P\{U(y,\lambda')\overset{\otimes}{,}U(x,\lambda)\}P. \tag{180}$$

To see this, we write both sides in terms of indices:

$$
\begin{aligned}
\{U(x,\lambda)\overset{\otimes}{,}U(y,\lambda')\}_{ijkl} &= \{U(x,\lambda)_{ik},U(y,\lambda')_{jl}\} \quad \text{and}\\
(-P\{U(y,\lambda')\overset{\otimes}{,}U(x,\lambda)\}P)_{ijkl} &= -P_{ijpq}\{U(y,\lambda')\overset{\otimes}{,}U(x,\lambda)\}_{pqrs}P_{rskl}\\
&= -\delta_{iq}\delta_{jp}\{U(y,\lambda')_{pr},U(x,\lambda)_{qs}\}\delta_{rl}\delta_{sk}=-\{U(y,\lambda')_{jl},U(x,\lambda)_{ik}\}
\end{aligned} \tag{181}
$$

Thus, (180) is equivalent to (179). We may use the permutation operator to express the anti-symmetry of the r-matrix. On the one hand, the LHS is

$$\{U(x,\lambda)\overset{\otimes}{,}U(y,\lambda')\}=[r(\lambda-\lambda'),U(x,\lambda)\otimes I+I\otimes U(x,\lambda')]\delta(x-y). \tag{182}$$

On the other hand, the RHS is

$$
\begin{aligned}
-P\{U(y,\lambda')\overset{\otimes}{,}U(x,\lambda)\}P &= -P[r(\lambda'-\lambda),U(y,\lambda')\otimes I+I\otimes U(y,\lambda)]\delta(x-y)P\\
&= [-Pr(\lambda'-\lambda)P,P(U(y,\lambda')\otimes I+I\otimes U(y,\lambda))P]\delta(x-y)\\
&= [-Pr(\lambda'-\lambda)P,U(x,\lambda)\otimes I+I\otimes U(x,\lambda')]\delta(x-y).
\end{aligned} \tag{183}
$$

So the PB is antisymmetric provided the r-matrix satisfies $Pr(\lambda'-\lambda)P=-r(\lambda-\lambda')$. When written in components this reduces to (178). Indeed, the RHS is $-r(\lambda-\lambda')_{ikpl}$ while the LHS is

$$(Pr(\lambda'-\lambda)P)_{ikpl}=P_{ikab}r(\lambda'-\lambda)_{abcd}P_{cdpl}=\delta_{ib}\delta_{ka}r(\lambda'-\lambda)_{abcd}\delta_{cl}\delta_{dp}=r(\lambda'-\lambda)_{kilp}. \tag{184}$$

Subscript Notation: Here we write the fundamental PBs as

$$\{U(x,\lambda)\overset{\otimes}{,}U(y,\lambda')\}=[r(\lambda-\lambda'),U_1(\lambda)+U_2(\lambda')]\delta(x-y)=[r_{12},U_1(\lambda)+U_2(\lambda')]\delta(x-y). \tag{185}$$

Here U_1 and U_2 are defined as:

$$U_1(\lambda)=U(\lambda)\otimes I \quad \text{and} \quad U_2(\lambda')=I\otimes U(\lambda') \tag{186}$$

For $i,j=1,2$ we define r_{ij} as: $r_{ij}=r(\lambda_i-\lambda_j)$. Here $\lambda_1=\lambda$ and $\lambda_2=\lambda'$. The LHS of (180) is $\{U(x,\lambda)\otimes,U(y,\lambda')\}$ while the RHS of (180) is given by

$$
\begin{aligned}
-P\{U(y,\lambda')\otimes,U(x,\lambda)\}P &= -P[r(\lambda'-\lambda),U_1(\lambda')+U_2(\lambda)]P\\
&= -P[r_{21},U_1(\lambda')+U_2(\lambda)]P=[-Pr_{21}P,P(U_1(\lambda')+U_2(\lambda))P]\\
&= [-Pr_{21}P,U_1(\lambda)+U_2(\lambda')]
\end{aligned} \tag{187}
$$

Here, we used $P^2=I$ and $PU_1(\lambda')P=U_2(\lambda')$. Comparing, anti-symmetry of the PB, is ensured if

$$-Pr_{21}P=r_{12}. \tag{188}$$

6.2.2 Jacobi identity and Classical Yang-Baxter equation

To qualify as a Poisson bracket, the fundamental PBs must satisfy the Jacobi identity. When expressed in terms of the classical r-matrix, the Jacobi identity is satisfied if the r-matrix satisfies a quadratic equation, the celebrated classical Yang-Baxter equation

$$[r_{12}(\lambda-\lambda'), r_{23}(\lambda'-\lambda'')] + [r_{23}(\lambda'-\lambda''), r_{31}(\lambda''-\lambda)] + [r_{31}(\lambda''-\lambda), r_{12}(\lambda-\lambda')] = 0. \quad (189)$$

The successive commutators are obtained via cyclic permutations. Each term here is an operator on the three-fold tensor product of the auxiliary linear space $V \otimes V \otimes V$. By definition, $r_{ij}(a,b)$ acts non-trivially only on the i^{th} and j^{th} factors of the tensor product $V \otimes V \otimes V$. $r_{ij}(a,b)$ is defined by applying suitable permutations to the original r-matrix, which is assumed to act on the first and second spaces. Thus, $r_{12}(a,b) = r(a-b)$ while the anti-symmetry of the r-matrix is used to define

$$r_{21}(a,b) = P_{12} r_{12}(a,b) P_{12} = -r_{12}(b,a) = -r(b-a). \quad (190)$$

Here $P_{12} = P_{21}$ permutes the first two entries in a tensor product $P_{12}(u \otimes v \otimes w) = v \otimes u \otimes w$. More generally, we may define the permutations of the other pairs of entries $P_{23} = P_{32}$ and $P_{13} = P_{31}$ via

$$P_{23}(u \otimes v \otimes w) = u \otimes w \otimes v \quad \text{and} \quad P_{13}(u \otimes v \otimes w) = w \otimes v \otimes u. \quad (191)$$

We verify that any one of these permutations can be built from the other two in two different ways:

$$P_{23} = P_{12}P_{13}P_{12} = P_{31}P_{12}P_{31}, \quad P_{13} = P_{23}P_{12}P_{23} = P_{12}P_{23}P_{12}, \quad \text{and} \quad P_{12} = P_{31}P_{23}P_{31} = P_{23}P_{31}P_{23}. \quad (192)$$

These permutations are then used to define the other r-matrices:

$$\begin{aligned} r_{13}(a,b) &= P_{23} r_{12}(a,b) P_{23}, & r_{31}(a,b) &= P_{32} r_{21}(a,b) P_{32}, \\ r_{23}(a,b) &= P_{12} r_{13}(a,b) P_{12} \quad \text{and} & r_{32}(a,b) &= P_{21} r_{31}(a,b) P_{21}. \end{aligned} \quad (193)$$

Using the above relations among permutations, it is then verified that for NLSE,

$$r_{ij}(\lambda_i - \lambda_j) = \frac{\kappa P_{ij}}{2(\lambda_i - \lambda_j)} \quad (194)$$

Introducing $r_{ij} \equiv r_{ij}(\lambda_i - \lambda_j)$ with $\lambda_1 = \lambda, \lambda_2 = \lambda'$ and $\lambda_3 = \lambda''$, the CYBE takes the compact form

$$\text{CYBE} = [r_{12}, r_{23}] + [r_{23}, r_{31}] + [r_{31}, r_{12}] = 0. \quad (195)$$

Exercise: Verify that the NLSE r-matrix $r(\lambda, \lambda') = \kappa P/2(\lambda - \lambda')$ satisfies the CYBE. To see this, we use (194) and (192) to express the three commutators in the CYBE as (here $\lambda_{ij} = \lambda_i - \lambda_j$)

$$[r_{12}, r_{23}] = \frac{\kappa^2}{4}\frac{[P_{12}, P_{23}]}{\lambda_{12}\lambda_{23}} = \frac{\kappa^2}{4}\frac{[P_{31}, P_{12}]}{\lambda_{12}\lambda_{23}}, \quad [r_{23}, r_{31}] = \frac{\kappa^2}{4}\frac{[P_{23}, P_{31}]}{\lambda_{23}\lambda_{31}} \quad \text{and} \quad [r_{31}, r_{12}] = \frac{\kappa^2}{4}\frac{[P_{31}, P_{12}]}{\lambda_{31}\lambda_{12}}, \quad (196)$$

so that

$$[r_{12}, r_{23}] + [r_{31}, r_{12}] = \frac{\kappa^2}{4}\frac{1}{\lambda_{12}}\left(\frac{1}{\lambda_{23}} + \frac{1}{\lambda_{31}}\right)[P_{31}, P_{12}] = \frac{\kappa^2}{4}\frac{[P_{12}, P_{31}]}{\lambda_{23}\lambda_{31}}. \quad (197)$$

Finally, using $P_{12}P_{13} = P_{13}P_{23}$ and $P_{23}P_{13} = P_{13}P_{12}$,

$$[r_{12}, r_{23}] + [r_{23}, r_{31}] + [r_{31}, r_{12}] = \frac{\kappa^2}{4\lambda_{23}\lambda_{31}}([P_{12} + P_{23}, P_{31}]) = 0 \quad (198)$$

6.3 Monodromy and conserved quantities in involution for periodic BCs

The ultralocality of the FPBs among U's, along with the auxiliary linear equations

$$\partial_x T(x,y;\lambda) = U(x,\lambda)T(x,y;\lambda) \quad \text{and} \quad \partial_y T(x,y;\lambda) = -T(x,y;\lambda)U(y,\lambda). \tag{199}$$

leads to a commutator expression for the fundamental PBs between transition matrix elements

$$\{T(x,y;\lambda) \mathbin{\overset{\otimes}{,}} T(x,y;\mu)\} = [r(\lambda-\mu), T(x,y;\lambda)\otimes T(x,y;\mu)]. \tag{200}$$

To obtain this, use is made of the Leibnitz rule and the functional derivative

$$\frac{\delta T_{ab}(x,y;\lambda)}{\delta U_{jk}(z,\lambda)} = T_{aj}(x,z,\lambda)T_{kb}(z,y,\lambda). \tag{201}$$

Use $\operatorname{tr} A\otimes B = \operatorname{tr} A \operatorname{tr} B$ to deduce that the traces of transition matrices are in involution

$$\{\operatorname{tr} T(x,y;\lambda),\ \operatorname{tr} T(x,y;\mu)\} = 0. \tag{202}$$

However, $\operatorname{tr} T(x,y;\lambda)$ is generally not conserved as $\partial_t T(x,y) = V(x)T(x,y) - T(x,y)V(y)$ (155). As noted in §5.2, for periodic BCs, the monodromy matrix $T_L(t) = T(L,-L;t)$ evolves via a Lax equation $\partial_t T_L(\lambda) = [V(L), T_L]$ so that $F_L(\lambda) = \operatorname{tr} T_L(\lambda)$ is conserved. Thus it follows that the traces of the monodromy matrix for distinct values of the spectral parameter are in involution. The coefficients in a Laurent expansion of $F_L(\lambda)$ this furnish an infinite sequence of conserved quantities in involution. However, these integrals of motion may not be local functionals of ψ and ψ^*. It turns out that $P_L(\lambda) = \arccos(\frac{1}{2}\operatorname{tr} T_L(\lambda))$ is a convenient generating function for local integrals of motion.

For bosons moving in the interval $-a \le x \le a$ with periodic boundary conditions, the first four integrals of motion are

$$\begin{gathered} N = \textstyle\int_{-a}^{a} |\psi|^2\,dx, \quad P = \int_{-a}^{a} \Im\psi^*\psi_x\,dx, \quad E = \int_{-a}^{a} (|\psi_x|^2 + \kappa|\psi|^4)\,dx \\ \text{and} \quad Q = \textstyle\int_{-a}^{a} \left[\psi^*\psi_{xxx} - \kappa|\psi|^2\,(\psi\psi_x^* + 4\psi^*\psi_x)\right]\,dx. \end{gathered} \tag{203}$$

The conserved quantities N, P and E represent the number of bosons, their total momentum and energy.

6.4 Quantum NLS: Transition matrix, quantum R-matrix and RTT relation

We pass to the quantum theory by replacing the classical fields $\psi(x), \psi^*(x)$ by field operators $\hat{p}hi(x)$ and $\hat{p}si^\dagger(x)$ and the classical PBs by the canonical bosonic equal-time commutation relations

$$[\hat\psi(x), \hat\psi(y)] = [\hat\psi(x), \hat\psi(y)] = 0 \quad \text{and} \quad [\hat\psi(x), \hat\psi^\dagger(y)] = \hbar\delta(x-y), \quad . \tag{204}$$

Auxiliary linear problem: In the quantum theory, the connection matrices U, V (168) that appear in the zero-curvature representation become operator-valued 2×2 matrices. For instance,

$$\hat U(x,\lambda) = \sqrt{\kappa}(\hat\psi^\dagger\sigma_+ + \hat\psi\sigma_-) + \lambda\frac{\sigma_3}{2i}. \tag{205}$$

Upon normal ordering, the space part of the auxiliary linear equation is

$$\partial_x \hat F(x) =: \hat U(x,\lambda)\hat F(x) := \sqrt{\kappa}(\sigma_+\hat\psi^\dagger\hat F + \sigma_-\hat F\hat\psi) + \lambda\frac{\sigma_3}{2i}\hat F. \tag{206}$$

Quantum r-matrix: It is convenient to use the classical r-matrix to define a quantum r-matrix which in this case is independent of $\hat{\psi}, \hat{\psi}^\dagger$:

$$R(\lambda) = I - i\hbar r(\lambda) \tag{207}$$

Quantum fundamental commutation relations: The quantum version of the fundamental PBs of $U(x,\lambda)$ may be expressed as

$$R(\lambda-\lambda')\hat{L}_1(x,\lambda,\lambda') = \hat{L}_2(x,\lambda,\lambda')R(\lambda-\lambda') \tag{208}$$

Here

$$\hat{L}_1(x,\lambda,\lambda') = \hat{U}(x,\lambda)\otimes I + I\otimes\hat{U}(x,\lambda') + \hbar\kappa\sigma_-\otimes\sigma_+ \quad \text{and} \quad \hat{L}_2(x,\lambda,\lambda') = \hat{U}(x,\lambda)\otimes I + I\otimes\hat{U}(x,\lambda') + \hbar\kappa\sigma_+\otimes\sigma_-. \tag{209}$$

Transition matrix: Next we introduce the operator-valued transition matrix

$$\hat{T}(x,y,\lambda) =: P\exp\int_y^x \hat{U}(z,\lambda)\,dz : \tag{210}$$

As in the classical case, we have the 'initial condition' $\hat{T}(x,x,\lambda) = I$ and the composition law

$$\hat{T}(x,y,\lambda)\hat{T}(y,z,\lambda) = \hat{T}(x,z,\lambda) \quad \text{for} \quad x > y > z. \tag{211}$$

The transition matrix is the solution of the auxiliary linear equations:

$$\partial_x\hat{T}(x,y,\lambda) =: \hat{U}(x,\lambda)\hat{T}(x,y,\lambda) : \quad \text{and} \quad \partial_y\hat{T}(x,y,\lambda) = - : \hat{T}(x,y,\lambda)U(y,\lambda) : . \tag{212}$$

For NLS, $\hat{T}$ is a linear operator on $V = \mathbb{C}^2$. On $V\otimes V$ we can define two obvious actions of T: either on the first or second factors:

$$\hat{T}_1 = \hat{T}\otimes I \quad \text{and} \quad \hat{T}_2 = I\otimes T. \tag{213}$$

RTT relation: The RTT relation is the quantum analogue of the fundamental PBs among transition matrix elements. It is a relation between two operators on $V\otimes V$:

$$R(\lambda-\lambda')T_1(x,y,\lambda)T_2(x,y,\lambda') = T_2(x,y,\lambda')T_1(x,y,\lambda)R(\lambda-\lambda'). \tag{214}$$

When $\hbar\to 0$ it reduces to

$$\{T(x,y,\lambda), T(x,y,\lambda')\} = [r(\lambda-\lambda'), T(x,y,\lambda)\otimes T(x,y,\lambda')]. \tag{215}$$

6.5 Quantum Yang-Baxter equation

The QYBE is a relation that the quantum R-matrix must obey, just as the CYBE is a condition on the classical r-matrix that ensures that the fundamental PBs obey the Jacobi identity. It may be obtained as a consistency condition by eliminating the transition matrices from the RTT relation. Recall that the CYBE (189) was a relation among operators on $V\otimes V\otimes V$. To obtain the quantum analogue, we introduce the three transition operators

$$\hat{T}_1 = \hat{T}\otimes I\otimes I, \quad \hat{T}_2 = I\otimes\hat{T}\otimes I \quad \text{and} \quad \hat{T}_3 = I\otimes I\otimes\hat{T} \tag{216}$$

as well as the quantum R-matrices $R_{ij}(\lambda-\mu)$ for $i,j=1,2,3$ that act on the ij subspace (based on the corresponding classical r-matrices). Consequently, R_{12} commutes with T_3 as they act on different spaces, etc. The QYBE is the cubic relation

$$R_{12}(\lambda-\lambda')R_{13}(\lambda-\lambda'')R_{23}(\lambda'-\lambda'')=R_{23}(\lambda'-\lambda'')R_{13}(\lambda-\lambda'')R_{12}(\lambda-\lambda'). \tag{217}$$

Notice that the order of the R's on the right is opposite to that on the left. We may view it as a sufficient condition for the consistency of the RTT relation.

Sketch of proof: To obtain (217) we start with the LHS and apply it to $\hat{T}_1\hat{T}_2\hat{T}_3$ and use the RTT relation thrice to get

$$\begin{aligned}&R_{12}(\lambda-\lambda')R_{13}(\lambda-\lambda'')R_{23}(\lambda'-\lambda'')\hat{T}_1(x,y,\lambda)\hat{T}_2(x,y,\lambda')\hat{T}_3(x,y,\lambda'')\\&=\hat{T}_3(x,y,\lambda'')\hat{T}_2(x,y,\lambda'')\hat{T}_1(x,y,\lambda)R_{12}(\lambda-\lambda')R_{13}(\lambda-\lambda'')R_{23}(\lambda'-\lambda'')\end{aligned} \tag{218}$$

Each time we move an R_{ij} through T_iT_j, the order of the T's is reversed. Similarly, we apply the RHS of (217) to the same product of transition operators to get

$$\begin{aligned}&R_{23}(\lambda'-\lambda'')R_{13}(\lambda-\lambda'')R_{12}(\lambda-\lambda')\hat{T}_1(x,y,\lambda)\hat{T}_2(x,y,\lambda')\hat{T}_3(x,y,\lambda'')\\&=\hat{T}_3(x,y,\lambda'')\hat{T}_2(x,y,\lambda'')\hat{T}_1(x,y,\lambda)R_{23}(\lambda'-\lambda'')R_{13}(\lambda-\lambda'')R_{12}(\lambda-\lambda').\end{aligned} \tag{219}$$

If we denote the LHS and RHS of (217) by R_F and R_B (for forward and backward) then we have

$$R_FT_1T_2T_3=T_3T_2T_1R_F \quad \text{and} \quad R_BT_1T_2T_3=T_3T_2T_1R_B \tag{220}$$

Now assuming R_F and R_B are invertible (for NLS, they are made from permutations (207), so this should be true), we get

$$R_FT_1T_2T_3R_F^{-1}=T_3T_2T_1=R_BT_1T_2T_3R_B^{-1} \quad \text{or} \quad R_B^{-1}R_FT_1T_2T_3(R_B^{-1}R_F)^{-1}=I. \tag{221}$$

So $R_B^{-1}R_F$ must commute with arbitrary transition matrix products. For this to be true, it is sufficient that $R_B^{-1}R_F$ be a multiple of the identity $R_F=\gamma R_B$. For NLS, using the formula $R=I-i\hbar r$ and comparing coefficients of powers of $\hbar$ we find $\gamma=1$.

We may interpret (218) and (219) as expressing the equality of two different ways of transforming $\hat{T}_1\hat{T}_2\hat{T}_3$ to $\hat{T}_3\hat{T}_2\hat{T}_1$. The two ways differ in the order in which pair transpositions are performed to reverse the order of T's. This can be depicted diagrammatically as in Fig. 4.

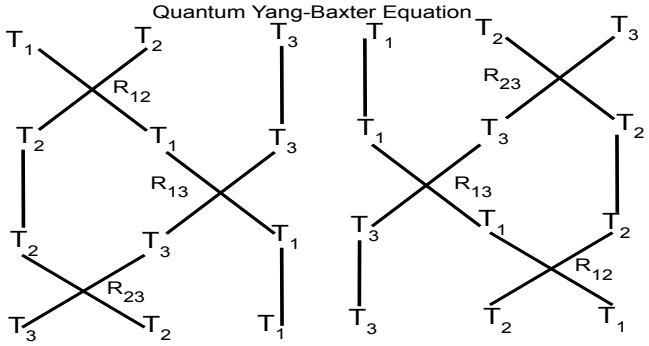

Figure 4: Quantum Yang-Baxter equation from equality of two ways of taking $T_1T_2T_3\to T_3T_2T_1$.

The NLSE quantum R-matrix satisfies the QYBE. For NLSE, $R(\lambda)=I-i\hbar r(\lambda)$ (207) where $r(\lambda)$ is the classical r-matrix. Collecting the terms in the QYBE, $R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$ at order

$\hbar^0, \hbar, \hbar^2$ and $\hbar^3$, we get four conditions. The condition at $\hbar^0$ is automatic, it simply says $I = I$. The condition at the next order is also automatically satisifed: $r_{12} + r_{13} + r_{23} = r_{23} + r_{13} + r_{12}$. At order $\hbar^2$ we get the CYBE for r. Finally order $\hbar^3$, we get the condition $r_{12}r_{13}r_{23} = r_{23}r_{13}r_{12}$. Using $r_{ij} = (\kappa/2)P_{ij}/(\lambda_i - \lambda_j)$ this is satisfied provided $P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12}$. It is easily verified using (192) that both sides are equal to P_{13}. Thus, the NLSE quantum R-matrix satisfies the QYBE.

7 Survey of some other developments

7.1 Bäcklund transformations

Roughly, Bäcklund transformations provide a way of generating new solutions of a PDE from known solutions of the same or a related PDE [14]. PDEs solvable by the inverse scattering transform usually admit a Bäcklund transformation.

Laplace's equation. The Laplace equation in two variables $\Delta u = (\partial_x^2 + \partial_y^2)u = 0$ is perhaps the simplest equation admitting an (auto-) Backlund transformation. Recall that the real and imaginary parts of an analytic function $f(x, y) = u(x, y) + iv(x, y)$ are both harmonic: $\Delta u = \Delta v = 0$. This allows us to view the Cauchy-Riemann (CR) equations $u_x = v_y$ and $u_y = -v_x$ as a Bäcklund transformation. Indeed, if we have a harmonic function $v(x, y)$ we may regard it as the imaginary part of some analytic function and view the CR equations as equations to determine the real part u, which is then guaranteed to be harmonic. For example, given the harmonic function $v = x^2 - y^2$, the CR equations become

$$u_x = -2y \quad \text{and} \quad u_y = -2x, \tag{222}$$

with the solution $u = -2xy$ upto an additive constant. u is evidently harmonic and the corresponding analytic function is $f(z) = iz^2$ where $z = x + iy$. We notice here that the CR equations are of one order less than the Laplace equation.

• More generally, suppose we have a pair of PDEs $P(u, u_t, u_x, \cdots) = 0$ and $Q(v, v_t, v_x, \cdots) = 0$ for functions u and v of two independent variables x, t. Then the relations $R_{1,2}(u, v, u_t, v_t, u_x, v_x, \cdots) = 0$ are said to be a Bäcklund transformation if given a v with $Q(v, \cdots) = 0$ the relations $R_{1,2}$ may be integrated to obtain a $u(x, t)$ such that $P(u, \cdots) = 0$. If P and Q are the same set of PDEs, then we call the relations an auto-Bäcklund transformation.

Liouville and wave equations. The Liouville equation (in 1+1 dimensional light-cone coordinates $x, t = x^0 \pm x^1$) is $u_{xt} = e^u$. It is related via a Bäcklund transformation to d'Alembert's linear wave equation $v_{xt} = 0$. Indeed, consider the relations

$$R_1 : u_x + v_x = \sqrt{2}e^{(u-v)/2} \quad \text{and} \quad R_2 : u_t - v_t = \sqrt{2}e^{(u+v)/2}. \tag{223}$$

Differentiating R_1 in t and using R_2 leads to $u_{xt} + v_{xt} = e^u$. If v satisfies the wave equation, then we see that u must satisfy the Liouville equation. What solution of the Liouville equation does $v \equiv 0$ generate? ANS: $u = -2\log(c - (x + t)/\sqrt{2})$.

Auto-Bäcklund for Sine-Gordon equation. Bäcklund discovered the transformations that bear his name in the context of the sine-Gordon equation in hyperbolic geometry. In light-cone coordinates, the sine-Gordon equation is $u_{xt} = \sin u$. For an arbitrary nonzero constant a, Bianchi's form of the Bäcklund relations are the pair of equations

$$\frac{1}{2}(u + v)_x = a\sin((u - v)/2) \quad \text{and} \quad \frac{1}{2}(u - v)_t = \frac{1}{a}\sin((u + v)/2). \tag{224}$$

Differentiating the first in t and using the second we get

$$\frac{1}{2}(u+v)_{xt} = \cos\frac{u-v}{2}\sin\frac{u+v}{2}. \tag{225}$$

Similarly, differentiating the second in x and using the first,

$$\frac{1}{2}(u-v)_{tx} = \cos\frac{u+v}{2}\sin\frac{u-v}{2}. \tag{226}$$

Adding and subtracting and using $\sin(a+b) = \sin a\cos b + \cos a\sin b$, we find that both u and v must satisfy the SG equation. Thus (224) may be regarded as an auto-Bäcklund transformation for SG. For example, we may exploit this transformation to generate a non-trivial solution to SG starting from the trivial solution $v \equiv 0$. Indeed, in this case, the Bäcklund relations become

$$u_x = 2a\sin\frac{u}{2} \quad \text{and} \quad u_t = \frac{2}{a}\sin\frac{u}{2}. \tag{227}$$

These may be integrated to yield

$$2ax = 2\log|\tan\frac{u}{4}| + f(t) \quad \text{and} \quad \frac{2t}{a} = 2\log|\tan\frac{u}{4}| + g(x) \tag{228}$$

where f and g are arbitrary functions of integration. This leads to the solution

$$u = 4\arctan\left(\kappa e^{ax+t/a}\right) \tag{229}$$

where κ is a constant of integration. This solution is called the sine-Gordon kink for $a > 0$ and the anti-kink for $a < 0$. Plot it to find out why.

KdV-mKdV equations. The Miura transform can be viewed as a Bäcklund transformation between the KdV and mKdV equations. Recall that if v satisfies the mKdV equation $v_t - 6v^2v_x + v_{3x} = 0$, then the Miura transform $u = v^2 + v_x$ must satisfy the KdV equation $u_t - 6u6_x + u_{xxx} = 0$. However, in this case, the mKdV equation is no easier to solve than the KdV equation, so that the Miura transform is not very useful in generating solutions to KdV.

The Wahlquist-Estabrook relations for KdV provide a more practical auto-Bäcklund transformation for KdV. Here, one begins by verifying that if v satisfies the mKdV with advecting velocity boosted by λ,

$$v_t - 6(v^2+\lambda)v_x + v_{3x} = 0, \tag{230}$$

then the u given by the boosted Miura transformation $u = \lambda + v^2 + v_x$ satisfies the usual KdV equation $u_t - 6uu_x + u_{3x} = 0$. This is verified using the formulae:

$$\begin{aligned} u_t &= 12v^3v_x + 12\lambda vv_x - 2vv_{3x} + 12vv_x^2 + 6v^2v_{xx} + 6\lambda v_{xx} - v_{4x}, \quad u_x = 2vv_x + v_{xx} \\ 6uu_x &= 6\left[2\lambda vv_x + \lambda v_{xx} + 2v^3v_x + v^2v_{xx} + 2vv_x^2 + v_xv_{xx}\right], \quad u_{3x} = 6v_xv_{xx} + 2vv_{3x} + v_{4x}. \end{aligned} \tag{231}$$

Next, we notice that if v satisfies mKdV (230) then so does $-v$. Thus, for given v and λ we define two functions via boosted Miura transforms:

$$u_1 = \lambda + v^2 + v_x \quad \text{and} \quad u_2 = \lambda + v^2 - v_x \tag{232}$$

whose sum and difference are

$$u_1 + u_2 = 2(\lambda + v^2) \quad \text{and} \quad u_1 - u_2 = 2v_x. \tag{233}$$

Here on, it is convenient to introduce velocity potentials $\phi_{1,2}$ such that $u_{1,2} = \partial_x \phi_{1,2}$. Then

$$(\phi_1 + \phi_2)_x = 2(\lambda + v^2) \quad \text{and} \quad \phi_1 - \phi_2 = 2v \tag{234}$$

where we have absorbed a constant of integration into the definition of ϕ_i. The first of these equations is the space part of the Bäcklund relations. The time part is obtained by writing the boosted mKdV (230) in terms of ϕ_i using (234):

$$(\phi_1 - \phi_2)_t + 3(\phi_{1x}^2 - \phi_{2x}^2) + (\phi_1 - \phi_2)_{3x} = 0. \tag{235}$$

Together these may be viewed as defining an auto-Bäcklund transformation for the KdV equation, now regarded as an equation for the velocity potential. In other words, if ϕ_2 is a solution of

$$\phi_{xt} - 6\phi_x \phi_{xx} + \phi_{4x} = 0, \tag{236}$$

then ϕ_1 obtained from ϕ_2 by solving the Bäcklund relations is also a solution of (236). For instance, if we begin with the trivial solution $\phi_2 \equiv 0$, then it turns out that the resulting ϕ_1 corresponds to the soliton solution of KdV. Indeed, putting $\phi_2 = 0$ in the space part of the Bäcklund relations gives

$$\phi_{1x} = 2\lambda + \frac{1}{2}\phi_1^2 \tag{237}$$

which upon integration for $\lambda = -\kappa^2 < 0$ gives (assuming $|\phi_1| < 2\kappa$ is bounded)

$$\phi_1 = -2\kappa \tanh(\kappa x + f(t)) \tag{238}$$

where $f(t)$ is an arbitrary function of time. To find $f(t)$ we use the time part of the Bäcklund relation

$$\phi_{1t} - 2\phi_{1x}^2 + \phi_{1xxx} = 0. \tag{239}$$

Using (237) twice, we may write the third derivative of ϕ_1 in terms of the first derivative:

$$\phi_{1xxx} = \frac{1}{2}(\phi_1^2)_{xx} = \phi_{1x}^2 + \phi_1^2 \phi_{1x}. \tag{240}$$

Thus the time part becomes

$$\phi_{1t} - \phi_{1x}\left(\phi_{1x} - \frac{1}{2}\phi_1^2\right) = 0 \quad \text{or} \quad \phi_{1t} + 4\kappa^2 \phi_{1x} = 0, \tag{241}$$

which is the 1d linear first order wave equation with speed $4\kappa^2$. It has the general d'Alembert solution

$$\phi_1 = g(x - 4\kappa^2 t) \tag{242}$$

where g is an arbitrary function. To be consistent with (238) we must have $f(t) = -4\kappa^3 t - \kappa x_0$ where x_0 is an arbitrary constant. Thus, the auto-Bäcklund transformation of the trivial solution gives the solitary wave:

$$\phi_1 = -2\kappa \tanh(\kappa(x - 4\kappa^2 t - x_0)) \quad \text{or} \quad u_1 = -2\kappa^2 \text{sech}^2\left[\kappa(x - x_0 - 4\kappa^2 t)\right]. \tag{243}$$

Note that if $|\phi|_1 > 2\kappa$ then the above procedure leads to an unbounded (singular) solution

$$\phi_1 = -2\kappa \coth\left[\kappa(x - 4\kappa^2 t - x_0)\right] \tag{244}$$

There is a clever algebraic way (without need for further integrations) [14] to use this Bäcklund transformation to generate 2-soliton solutions using the above bounded and unbounded solutions. More generally, one obtains n-soliton solutions this way.

7.2 Painlevé property and conjecture

ODEs with the Painlevé property: The Painlevé property is a feature of linear ODEs that however is also shared by certain special classes of nonlinear ODEs. ODEs with the Painlevé property have been found to be associated with many integrable field theories. In fact, it is conjectured that a PDE is solvable by the IST if every exact reduction to an ODE leads to an ODE of Painlevé type.

The linear ODE $w'(z) = w$ has the general solution $w(z) = Ae^z$. The latter has an essential singularity at $z = \infty$, for any value of the integration constant A. Similarly, the general solution $w = A/z$ of the linear equation $zw' + w = 0$ has a pole at $z = 0$, again for any value of A. In fact, it is generally true that singularities of linear ODEs are not movable, i.e., they occur at the same point(s) independent of the values of constants of integration. On the other hand, the Riccati-like nonlinear ODE $w' + w^2 = 0$ has the general solution $w = 1/(z - z_0)$. The latter has a simple pole at z_0, which however, moves around the complex plane as the constant of integration z_0 is varied. Other such examples include the nonlinear ODES $w' = 1/w$ [which has a movable square-root branch point $w = \sqrt{2(z - z_0)}$] and $w' = e^{-w}$ [which has a movable logarithmic branch point $w = \log(z - z_0)$]. Thus, it it clear that solutions to nonlinear ODEs can have movable singularities. Of these singularities, it is useful to distinguish between poles on the one hand and singularities like branch-cuts and essential singularities on the other. The main distinction is that a meromorphic function (one whose singularities are poles) is single-valued, while a function with branch-cuts or essential singularities is not. Remarkably, there are some special nonlinear ODEs whose solutions have no *movable* singularities *other* than poles. We thus define a *critical point* of an ODE as a point where the solution has a singularity (fails to be analytic) which is not a pole. We will say that an ODE has the Painlevé property if it has no movable critical points. The Painlevé property thus generalizes this feature of linear ODEs.

In fact, prior to Painlevé's work, which concerned 2nd order ODEs, Fuchs had shown in 1884 that the only first order ODEs of the form $w' = F(z, w)$ (where F is rational in w and analytic in z) whose solutions do not have any movable critical points are those of the generalized Riccati-type, where $F = a(z) + b(z)w + c(z)w^2$ is a quadratic polynomial in w with analytic coefficients a, b and c. The example $w' = 1/w$ with solution $w = \sqrt{2(z - z_0)}$ shows that a movable branch cut can arise if F is not a quadratic polynomial in w.

The case of 2nd order ODEs $w'' = F(z, w, w')$ where F is rational in w and w' and analytic in z, was investigated by Painlevé and his student Gambier from 1900. They showed that there are 50 different cases with no movable critical points. Of these 50 equations, only 44 could be solved in terms of elementary or elliptic functions. The remaining 6 are the Painlevé equations with the standard forms [26]

$$\begin{aligned}
&\text{(PI)}: \; w'' = 6w^2 + z, \quad \text{(PII)}: \; w'' = 2w^3 + zw + \alpha, \quad \text{(PIII)}: \; w'' = \tfrac{w'^2}{w} - \tfrac{w'}{z} + \tfrac{\alpha w^2 + \beta}{z} + \gamma w^3 + \tfrac{\delta}{w} \\
&\text{(PIV)}: ww'' = \tfrac{1}{2}(w')^2 + \beta + 2(z^2 - \alpha)w^2 + 4zw^3 + \tfrac{3}{2}w^4, \\
&\text{(PV)}: w'' = \left(\tfrac{1}{2w} + \tfrac{1}{w-1}\right) w'^2 - \tfrac{w'}{z} + \tfrac{(w-1)^2}{z^2}\left(\alpha w + \tfrac{\beta}{w}\right) + \tfrac{\gamma w}{z} + \delta\tfrac{w(w+1)}{w-1} \\
&\text{(PVI)}: w'' = \tfrac{1}{2}\left(\tfrac{1}{w} + \tfrac{1}{w-1} + \tfrac{1}{w-z}\right) w'^2 - \left(\tfrac{1}{z} + \tfrac{1}{z-1} + \tfrac{1}{w-z}\right) w' \\
&\qquad + \tfrac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \beta\tfrac{z}{w^2} + \gamma\tfrac{(z-1)}{(w-1)^2} + \delta\tfrac{z(z-1)}{(w-z)^2}\right)
\end{aligned} \tag{245}$$

where $\alpha, \beta, \gamma, \delta$ are complex constants. For generic values of these constants, PI-PVI cannot be reduced to simpler equations and their solutions define new special functions called the Painlevé transcendents.

- Not much is known about third order ODEs with the Painlevé property.

The Painlevé conjecture: relates evolutionary PDEs that can be solved by the IST and ODEs of Painlevé type. It states that a nonlinear PDE is solvable by the IST if every ODE obtained from it by exact

reduction possesses the Painlevé property. The conjecture was formulated by Ablowitz, Ramani and Segur [3] and an argument for why it may be true was provided by McLeod and Olver [35]. There is much evidence that the conjecture may hold under suitable hypotheses[26] and so can be used as a test for integrability via the IST. It is also believed that the Painlevé property of ODEs obtained by exact reduction is a reflection of a deeper single-valuedness property of solutions to the original integrable PDE; the Painlevé property has also been related directly to the PDE [50].

Here, all we will do is illustrate the idea of an exact reduction of a PDE to an ODE. Such a reduction may be obtained, for instance, by looking for traveling wave or similarity solutions. For instance, in the case of KdV, our search in §4.2 for similarity solutions of the form $u(x,t) = -(3t)^{-2/3} f(\eta)$ where the scaling variable $\eta = x(3t)^{-1/3}$ led us to the ODE $f''' + (6f - \eta)f' - 2f = 0$ (42). Upon making the substitution $f = \lambda w' - w^2$, this reduces to $w'' = \eta w + 2w^3$ which we recognize as the second Painlevé equation with $\alpha = 0$. On the other hand, the ODE describing traveling wave solutions of the KdV equation $u(x,t) = f(x - ct)$ is $f'' - 3f^2 - cf = A$. This equation is simpler than the Painlevé equations and can be solved in terms of elliptic functions, which are known not to have singularities any worse than poles.

7.3 Instantons and Anti-self-dual Yang-Mills

• It is believed that any PDE solvable by IST arises as a reduction of the (anti-)self-dual Yang-Mills equations [15].

• **Instantons** are finite action solutions of the classical equations of motion in Euclidean space-time (i.e., imaginary time $\mathbb{R}^4$ with metric $\eta_{\mu\nu} = \text{diag}(1,1,1,1)$). The name (given by G. 't Hooft) derives from the fact that such a solution is localized in both space and Euclidean time. They are relevant to understanding tunneling across a barrier in quantum mechanics.

• Instantons may be viewed as the Euclidean field theory counterparts of solitary waves/solitons: localized finite energy solutions to the field equations. In 1+1 dimensions, kinks of the ϕ^4 or Sine-Gordon model are examples of solitons in relativistic field theories. In 2+1 dimensions, vortices (say in the O(2) Nonlinear Sigma model) provide examples of solitons. In 3+1 dimensions, examples of solitons include Skyrmions (in the SU(2) nonlinear sigma model). Solitons can owe their existence and stability to the competing effects of nonlinearity and dispersion, as in KdV. They could also owe their existence/stability to nontrivial topology (say of the configuration space of fields). In the latter case they are called topological solitons.

• **Euclidean Yang-Mills Theory:** We will be concerned with Euclidean pure Yang-Mills theory whose dynamical variable is an $G = SU(2)$ Lie algebra valued 1 form $A = A_\mu dx^\mu$. If T^a are a basis for the SU(2) Lie algebra, $A_\mu = A^a_\mu T^a$ with

$$T_a = \frac{i}{2}\sigma_a, \quad [T_a, T_b] = -\epsilon_{abc} T_c \quad \text{and} \quad \text{tr}\, T_a T_b = -\frac{1}{2}\delta_{ab} \quad \text{for} \quad a,b = 1,2,3. \tag{246}$$

The action is

$$S = -\int_{\mathbb{R}^4} \text{tr}\, F \wedge *F = -\frac{1}{2}\,\text{tr} \int F_{\mu\nu} F^{\mu\nu} d^4x = \frac{1}{4}\int F^a_{\mu\nu} F^{\mu\nu a} d^4x, \tag{247}$$

[26] Some restrictions would be necessary. For instance, the Dym equation $u_t = u^3 u_{3x}$ is solvable by the IST but does not possess the Painevé property.

where F is the field strength 2-form, also valued in the Lie algebra:

$$F = DA = dA + A \wedge A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{or} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{248}$$

We check that F automatically satisfies the Bianchi identity $DF = 0$ where $DF = dF + [A, F]$ is the gauge covariant derivative[27]. The Hodge dual $*$ is a map taking p forms to $4-p$ forms defined on basis forms by

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{\sqrt{|\det \eta|}}{(4-p)!} \epsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_4} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_4}. \tag{249}$$

In particular, $(*F)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$.

The resulting YM equations are $D * F = 0$. Finiteness of the action S is ensured if the fields decay as

$$F_{\mu\nu}(x) \to \mathcal{O}(1/r^3) \quad \text{and} \quad A_\mu(x) \to -\partial_\mu g g^{-1} + \mathcal{O}(1/r^2) \quad \text{as} \quad r \to \infty. \tag{250}$$

Here $g : \mathbb{R}^4 \to G$ is a gauge transformation. Asymptotically, g defines a map from $S^3_\infty \to SU(2)$.

• **Anti-self-duality and instantons:** There is a remarkable relation between YM instantons and Anti-self dual connections. Four dimensional gauge fields possess a topological invariant, the integral of the 2nd Chern class

$$c_2 = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \operatorname{tr} F \wedge F \tag{251}$$

which is an integer.

• Interestingly, for $c_2 > 0$ the YM action S is bounded below by $8\pi^2 c_2$ i.e.,

$$S \geq 8\pi^2 c_2 \tag{252}$$

To see this, we note that since $F \wedge F = *F \wedge *F$ (check by writing in components) and $\operatorname{tr} F \wedge *F = \operatorname{tr} * F \wedge F$,

$$S = -\int_{\mathbb{R}^4} \operatorname{tr} F \wedge *F = -\frac{1}{2} \operatorname{tr} \int (F + *F) \wedge (F + *F) + \operatorname{tr} \int F \wedge F = -\frac{1}{2} \operatorname{tr} \int (F + *F) \wedge (F + *F) + 8\pi^2 c_2 \geq 8\pi^2 c_2. \tag{253}$$

Here we used $\operatorname{tr} T^a T^b = -\frac{1}{2} \delta^{ab}$, which ensures that the first term is non-negative and vanishes precisely when $F + *F = 0$. Thus, the bound (Bogomol'nyi bound) is saturated for anti-self-dual YM (ASDYM) fields, i.e., when $F = - * F$ which in components are the three equations

$$F_{12} = -F_{34}, \quad F_{13} = -F_{42} \quad \text{and} \quad F_{14} = -F_{23}. \tag{254}$$

On the other hand, if $c_2 \leq 0$ a similar argument shows that $S \leq -8\pi^2 c_2$ with the Bogomol'nyi bound saturated by self-dual YM fields $F = *F$.

In both the self-dual and anti-self-dual cases, the Bianchi identity $DF = 0$ implies that the YM equations $D * F = 0$ are satisfied. Thus, ASDYM fields which satisfy the BCs (250) are minima of the YM action and are therefore instantons. We also note that the ASDYM equations $F = - * F$, though nonlinear due to the nonabelian nature of G, are first order unlike the 2nd order YM equations $D * F = 0$. Thus, solving the ASD equations should be significantly simpler than solving the full YM equations.

[27] For a field ϕ in the fundamental representation of G, $(D_\mu \phi)^a = \partial_\mu \phi^a - \epsilon_{abc} A^b_\mu \phi^c$

Corrigan and Fairlie ansatz for exact solutions of ASD equations: An ansatz for the ASD gauge field A may be obtained by factorizing the space dependence from the internal degrees of freedom:

$$A = \sigma_{\mu\nu}\frac{\partial_\nu \rho}{\rho}dx^\mu \tag{255}$$

In order for A to be ASD $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ are chosen to be 4×4 SD matrices:

$$\sigma_{ab} = \epsilon_{abc}T_c \quad \text{and} \quad \sigma_{a4} = -\sigma_{4a} = T_a. \tag{256}$$

Note that $\sigma_{12} = \sigma_{34}$ etc. σ is SD in the sense that $\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\sigma_{\kappa\lambda} = \sigma_{\mu\nu}$. Moreover, the commutation relations among the $\sigma_{\mu\nu}$ are given by

$$[\sigma_{\mu\kappa}, \sigma_{\nu\lambda}] = -\delta_{\mu\nu}\sigma_{\kappa\lambda} + \delta_{\mu\lambda}\sigma_{\kappa\nu} + \delta_{\kappa\nu}\sigma_{\mu\lambda} - \delta_{\kappa\lambda}\sigma_{\mu\nu}. \tag{257}$$

It can be shown that the potential A satisfies the ASDYM equations iff $\Box\rho = 0$.

Jackiw-Nohl-Rebbi N instanton solutions: The 'fundamental' solution to the Laplace equation $\rho = r^{-2}$ (centered at $r = 0$) leads to a pure gauge A with $F = 0$. The N-instanton solutions are obtained by superposing $(N+1)$ fundamental solutions to the Laplace equation centered at $x_0, \cdots, x_N$:

$$\rho = \sum_{p=0}^{N}\frac{\lambda_p}{|x - x_p|^2}. \tag{258}$$

The limiting case, where one of the centers is at ∞ is related to the 't Hooft ansatz:

$$\rho = 1 + \sum_{p=1}^{N}\frac{\lambda_p}{|x - x_p|^2}. \tag{259}$$

Lax Pair for ASD fields: Here we consider a complexified Minkowski space $\mathbb{C}^4$ with coordinates $z, \tilde{z}, w, \tilde{w}$ and the flat metric $ds^2 = 2(d\tilde{z}dz - d\tilde{w}dw)$. Further, we choose the orientation given by the volume form

$$\omega = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}. \tag{260}$$

The space of self-dual ($\phi = *\phi$) two-forms is spanned by:

$$\omega_1 = dw \wedge dz, \quad \omega_2 = dw \wedge d\tilde{w} - dz \wedge d\tilde{z} \quad \text{and} \quad \omega_3 = d\tilde{z} \wedge d\tilde{w}. \tag{261}$$

Writing $D_w = \partial_w + A_w$ etc., we find that the ASD condition $F = -*F$ is equivalent to[28]

$$F \wedge \omega_i = 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{or} \quad F_{wz} = 0, \quad F_{w\tilde{w}} = F_{z\tilde{z}} \quad \text{and} \quad F_{\tilde{w}\tilde{z}} = 0. \tag{262}$$

The ASDYM equations (254) admit a zero curvature representation in terms of the covariant derivatives:

$$L = D_{\tilde{z}} - \lambda D_w \quad \text{and} \quad M = D_{\tilde{w}} - \lambda D_z \quad \text{with} \quad \lambda \in \{\mathbb{C} \cup \infty\}. \tag{263}$$

Using $F_{wz} = [D_w, D_z]$ etc., the curvature

$$R = [L, M] = [D_{\tilde{z}} - \lambda D_w, D_{\tilde{w}} - \lambda D_z] = F_{\tilde{z}\tilde{w}} - \lambda(F_{w\tilde{w}} - F_{\tilde{z}z}) + \lambda^2 F_{wz} \tag{264}$$

vanishes for every value of λ iff the ASDYM equations (262) are satisfied. Thus, we may view the ASDYM equations as arising as the compatibility condition for the overdetermined linear system

$$L\psi = 0 \quad \text{and} \quad M\psi = 0. \tag{265}$$

[28] In general, $\eta \wedge *\omega = \omega \wedge *\eta$. Thus, $F \wedge *\omega = \omega \wedge *F$. Using $a \wedge b = b \wedge a$ for two-forms and the self-duality condition on ω, this implies $F \wedge \omega = *F \wedge \omega$. Now, the anti-self-duality condition on F becomes $2F \wedge \omega = 0$.

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A Brief primer on Hamiltonian mechanics

Three ingredients go into the formulation of a classical Hamiltonian mechanical system: phase space, Poisson brackets (PBs) and the Hamiltonian. The phase space is a differentiable manifold M, whose points (say with coordinates ξ^i) are possible instantaneous states of the system; e.g., $M = \mathbb{R}^{2n}$ for a point particle moving in n dimensions where ξ may be taken to be the n Cartesian coordinates q^i and momenta p_j. Observables (or dynamical variables) are given by smooth real-valued functions $f(\xi)$ on phase space. These can include the components of position (q^2), momentum (p_3), angular momentum $(q^1p_2 - q^2p_1)$, kinetic $(\sum_j p_j^2/2m)$ and potential energy $(V(q))$. The observables form a commutative and associative algebra under point-wise products $(fg)(\xi) = f(\xi)g(\xi) = (gf)(\xi)$. However, the algebra of observables also carries a non-associative product that is a vestige of the possible lack of commutativity of observables in the quantum theory: a Poisson structure. This means that given any pair of observables f and g, their PB $\{f, g\}$ is again an observable. To be a PB, the bracket must satisfy four conditions for any observables f, g and h:

1. linearity $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$ for any real numbers a, b.
2. anti-symmetry: $\{f, g\} = \{g, f\}$,
3. the Leibnitz/product rule $\{f, gh\} = g\{f, h\} + \{f, g\}f$ and
4. the Jacobi identity $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$.

The physical meaning of these conditions will be clarified shortly. A manifold M with a Poisson structure is called a Poisson manifold. For the point particle, the PB is given by the familiar formula

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \right), \tag{266}$$

which is manifestly antisymmetric, satisfies the Leibnitz rule on account of involving only first order derivatives of f and g and may be verified to satisfy the Jacobi identity. The Poisson structure may be encoded in a second rank anti-symmetric tensor field. Given the coordinates ξ^i on phase space, the components of the Poisson tensor are $\{\xi^i, \xi^j\} = r^{ij}(\xi)$. In the above point particle case, say with $n = 1$ and $\xi = (q, p)$,

$$r = \begin{pmatrix} \{q, q\}, \{q, p\} \\ \{p, q\}, \{p, p\} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{267}$$

is a 2×2 antisymmetric tensor with constant components. The Leibnitz rule may be used to show that the PB acts as a derivation (first order differential operator) in each slot. For instance,

$$\{\xi^i, F\} = \sum_j \{\xi^i, \xi^j\}\frac{\partial F}{\partial \xi^j}. \tag{268}$$

Verify this for polynomial observables $F(\xi)$.

The last ingredient in the specification of a Hamiltonian system is a distinguished observable called the Hamiltonian H, which usually has the interpretation of energy. For a particle in a potential moving in Euclidean space $\mathbb{R}^n$, $H = (1/2m)\sum_i p_i^2 + V(q^1, \cdots, q^n)$.

Time evolution: With the above ingredients, one can define a natural notion of time evolution. Given any observable f, its infinitesimal evolution is determined by the first order ODE $\dot{f} = \{f, H\}$. We say that the Hamiltonian generates time evolution via the PB. For the particle, one verifies that this leads to the familiar equations of Hamilton:

$$\dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i}. \tag{269}$$

Given an initial state (phase point), Hamilton's equations in principle determine a phase trajectory, which is the curve $\xi(t)$ on phase space. Thus, the Hamiltonian generates a flow on the phase space. Replacing the Hamiltonian by any other observable, say F, we see that every observable generates a flow on the phase space. The infinitesimal change in the coordinate ξ^i under such a flow is $\xi^i \mapsto \eta^i = \xi^i + \delta\xi^i$ where $\delta\xi^i = \{\xi^i, F\}$. For a particle, the infinitesimal change in the position and momenta due to the flow generated by ϵF is given by

$$\delta q^i = \{q^i, \epsilon F\} = \epsilon\frac{\partial F}{\partial p_i} \quad \text{and} \quad \delta p_i = \{p^i, \epsilon F\} = -\epsilon\frac{\partial F}{\partial q_i}. \tag{270}$$

Here ϵ is a small book-keeping parameter which may be set to 1. Of particular interest to us are conserved quantities. These are special observables that are constant along trajectories: $dF/dt = 0$ (their values can vary continuously from trajectory to trajectory). It is readily seen then that F is conserved iff $\{F, H\} = 0$. Thus one utility of the PB is to check if an observable is conserved. Anti-symmetry then implies that the Hamiltonian is automatically conserved $\dot{H} = \{H, H\} = 0$. Poisson's theorem implies that the PB of a pair of conserved quantities is also conserved - verify this! Thus, in favorable cases, the PB may be used to generate new conserved quantities from existing ones.

• By a **canonical transformation** we mean a transformation of the phase space that preserves the PBs or equivalently, the Poisson tensor. Suppose $\eta^i = \eta^i(\xi)$ is a change of coordinates (diffeomorphism) of the phase space. It is said to be canonical if $\{\eta^i, \eta^j\} = r^{ij}(\eta)$ if $\{\xi^i, \xi^j\} = r^{ij}(\xi)$ for the same tensor, now evaluated at the new location. The flows generated on phase space by observables are the prime examples of canonical transformations. Indeed, working to order ϵ, one checks using anti-symmetry and the Jacobi identity that

$$\begin{aligned}
\{\eta^i, \eta^j\} &= \{\xi^i + \{\xi^i, \epsilon F\}, \xi^j + \{\xi^j, \epsilon F\}\} = \{\xi^i, \xi^j\} + \epsilon\{\{\xi^i, F\}, \xi^j\} + \epsilon\{\xi^i, \{\xi^j, F\}\} + \mathcal{O}(\epsilon^2) \\
&= r^{ij}(\xi) + \epsilon\{\{\xi^i, \xi^j\}, F\} + \mathcal{O}(\epsilon^2) = r^{ij}(\xi) + \delta r^{ij} + \mathcal{O}(\epsilon^2) \\
&= r^{ij}(\xi + \delta\xi) + \mathcal{O}(\epsilon^2) = r^{ij}(\eta) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{271}$$

Hamiltonian and canonical vector fields: The first order system of ODEs

$$\dot{\xi}^i = \{\xi^i, H\} = \{\xi^i, \xi^j\}\frac{\partial H}{\partial \xi^j} = r^{ij}\partial_j H \equiv V_H^i(\xi) \tag{272}$$

defines a vector field with components V_H^i on the phase space. It is called the Hamiltonian vector field. Phase trajectories are the integral curves of this vector field so we may also denote it $\frac{d}{dt}$. More generally,

the flow generated by any observable F defines a canonical vector field V_F on phase space, whose components are obtained by contracting the Poisson tensor with $\partial_j F$:

$$V_F = r^{ij}(\partial_j F)\frac{\partial}{\partial \xi^i}. \tag{273}$$

In the point particle example, check that we have

$$V_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{274}$$

Using Hamilton's equations, we see that the Hamiltonian vector field may also be written in the suggestive manner

$$V_H = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} = \frac{d}{dt}. \tag{275}$$

More generally, the canonical vector field associated with the observable $F(q,p)$ may may be written as

$$V_F = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i} = \frac{\partial q^i}{\partial s}\frac{\partial}{\partial q^i} + \frac{\partial p_i}{\partial s}\frac{\partial}{\partial p_i} = \frac{d}{ds} \tag{276}$$

where s (like time t for H) is the parameter along the integral curves of F.

Symplectic form: In general, the Poisson tensor is not an invertible matrix at any given location of the phase space. For instance, take $M = \mathbb{R}^3$ to be the 3d phase space of angular momenta of a rigid body with cartesian coordinates L^1, L^2, L^3 and the usual angular momentum PBs

$$\{L^i, L^j\} = \sum_{k=1}^{3} \epsilon^{ijk} L^k, \tag{277}$$

where ϵ^{ijk} is the totally anti-symmetric Levi-Civita tensor. The corresponding Poisson tensor

$$r^{ij}(L) = \epsilon^{ijk}L^k = \begin{pmatrix} 0 & L^3 & -L^2 \\ -L^3 & 0 & L^1 \\ L^2 & -L^1 & 0 \end{pmatrix} \tag{278}$$

does not have constant components and is moreover nowhere invertible as a matrix. Indeed, the 'radial' 1-form with components (L^1, L^2, L^3) is annihilated by r (and one checks that the kernel of r is one-dimensional). Such a Poisson tensor is said to be degenerate. In terms of the Poisson algebra of observables, it means that there must be an observable (Casimir) that Poisson commutes with all observables. In the case at hand, this Casimir is the familiar square of angular momentum $l^2 = (L^1)^2 + (L^2)^2 + (L^3)^2$ which satisfies $\{l^2, L^i\} = 0$ for $i = 1, 2, 3$. Such a Casimir is conserved irrespective of the Hamiltonian as it commutes with all the coordinates and hence with all observables: $\{l^2, H\} = 0$ for any choice of H. In the quantum theory, Casimirs correspond to super-selection rules. Classically, the implication is that the trajectories are confined to a level surface of l^2 determined by initial conditions. In this case, these level surfaces are evidently concentric 'angular momentum spheres' centered at the origin. Though the Poisson tensor cannot be inverted on the full tangent space at any point of phase space, it can be inverted if restricted to the level surfaces of l^2. Indeed, one checks by going to spherical polar coordinates, that when restricted to any such sphere of nonzero radius, the Poisson tensor is invertible and its inverse $\omega_{ij} = (r^{-1})_{ij}$ is an anti-symmetric 2×2 matrix field proportional to the area 2-form on the sphere. These concentric spheres on which the Poisson tensor can be inverted are called the symplectic leaves of the Poisson manifold M.

With this background, let us now consider a Hamiltonian system with a non-degenerate Poisson structure [obtained, if necessary, by restricting the dynamics from the full Poisson manifold to a symplectic leaf - common levels set of all the Casimirs]. Such a 'reduced' phase space must necessarily be even dimensional, since it can be shown that an odd dimensional anti-symmetric matrix has zero determinant[29]. Inverting the Poisson tensor we obtain a non-degenerate rank-two anti-symmetric covariant tensor field $\omega = r^{-1}$ which is called the symplectic 2-form $\omega = \frac{1}{2}\omega_{ij}d\xi^i \wedge d\xi^j$. In the case of the point particle, the usual Poisson structure on $M = \mathbb{R}^{2n}$ is invertible everywhere, and one finds $\omega = \sum_{i=1}^n dq^i \wedge dp_i$. For one degree of freedom, this is the familiar area element $\omega = dq \wedge dp$ on the phase plane. In this case, using $d^2 = 0$, one checks that $d\omega = 0$. In fact, in this case ω is in fact exact $\omega = d\alpha$ where the canonical 1-form $\alpha = \sum_i p_i dq^i$. More generally, one can show that the Jacobi identity on r implies that ω is closed, $d\omega = 0$. Thus, we define a symplectic manifold as an even dimensional manifold equipped with a closed and non-degenerate two form ω. In terms of ω, the canonical vector field associated with an observable F is given by

$$V_F^i = (\omega^{-1})^{ij}\partial_j F \quad \text{or equivalently} \quad dF(\cdot) = \omega(V_H, \cdot). \tag{279}$$

For a point particle moving on the real line, we have

$$\omega = dq \wedge dp = dq \otimes dp - dp \times dq, \quad dF = \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp \tag{280}$$

so that $V_F = -\frac{\partial F}{\partial q}\frac{\partial}{\partial p} + \frac{\partial F}{\partial p}\frac{\partial}{\partial q}$ satisfies the defining condition:

$$\omega(V_F, \cdot) = 0 + \frac{\partial F}{\partial p}dp(\cdot) - \left(-\frac{\partial F}{\partial q}dq(\cdot) + 0\right) = dF(\cdot). \tag{281}$$

B KdV Jost wavefunctions and scattering data

B.1 Jost scattering wavefunctions

To begin with, we consider scattering eigenfunctions of the Schrödinger-Lax operator of KdV

$$L\phi = -\phi'' + u\phi = k^2\phi. \tag{282}$$

For real $u(x)$ satisfying the Faddeev condition ($\int_{\mathbb{R}}(1+|x|)u(x)dx < \infty$ which implies that u is integrable and decays to zero at $\pm\infty$), the continuous spectrum consists of all $k^2 > 0$ while the discrete spectrum corresponds to $k = i\kappa_n$ for $1 \leq n \leq N$. For given $k^2 > 0$ the scattering states are doubly degenerate corresponding to asymptotically 'left' (e^{-ikx}) and 'right' moving (e^{ikx}) plane waves (alternatively, if ϕ is an eigenfunction with eigenvalue k^2, then so is its complex conjugate $\bar{\phi}$). For definiteness, we will take $k > 0$ and denote by G_k the 2d eigenspace of L with eigenvalue k^2. Now, we introduce two convenient bases of complex conjugate pairs of Jost functions (φ_1, φ_2) and (ψ_1, ψ_2) for G_k with definite asymptotic behavior as $x \to \mp\infty$:

$$\varphi_1 \to e^{-ikx} \quad \text{and} \quad \varphi_2 \to e^{ikx} \quad \text{as} \quad x \to -\infty. \tag{283}$$

[29] Suppose r is an odd dimensional anti-symmetric matrix $r^t = -r$, then $\det r = \det r^t = \det(-r) = (-1)^{\text{odd}\#}\det r = -\det r$. Thus $2\det r = 0$.

ϕ_2 corresponds to a unit amplitude plane wave coming in from $-\infty$ and φ_1 a unit amplitude left-moving wave at $-\infty$. Similarly, ψ_1 and ψ_2 are left- and right-moving waves of unit amplitude at ∞:

$$\psi_1 \to e^{-ikx} \quad \text{and} \quad \psi_2 \to e^{ikx} \quad \text{as} \quad x \to \infty. \tag{284}$$

Interestingly, φ_2 and ψ_2 are the complex conjugates of φ_1 and ψ_2. In fact, for real u, if $\varphi_1(x,k)$ satisfies (282), then so does $\bar{\varphi}_1(x,k)$[30]. Moreover, since $\varphi_1 \to e^{-ikx}$ as $x \to -\infty$, it follows that $\bar{\varphi}_1(x,k) \to e^{ikx}$ as $x \to -\infty$. Consequently, both $\bar{\varphi}_1$ and φ_2 satisfy the same Sturm-Liouville problem (282) with the same eigenvalue k^2 and have the same asymptotic behavior. Therefore, $\varphi_2(x,k) = \bar{\varphi}_1(x,k)$. Similarly, $\psi_2(x,k) = \bar{\psi}_1(x,k)$.

Now, since (ψ_1, ψ_2) is a basis for G_k, we can write φ_1 and φ_2 as the linear combinations $\varphi_i = T_{ij}\psi_j$, where T is a 2×2 *transition* matrix. Denoting the first row of T as $(a(k), b(k))$ and using the previous result, we have

$$\varphi_1 = a(k)\psi_1 + b(k)\bar{\psi}_1 \tag{285}$$

Taking a complex conjugate,

$$\bar{\varphi}_1 = \varphi_2 = \bar{b}\psi_1 + \bar{a}\bar{\psi}_1. \tag{286}$$

In other words, the second row of T must be $(\bar{b}, \bar{a})$ and $T = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$. Henceforth, we will drop the subscripts 1 and write $\varphi_{1,2} = (\varphi, \bar{\varphi})$ and $\psi_{1,2} = (\psi, \bar{\psi})$. To interpret a and b we examine the asymptotic behaviors of $\varphi = a\psi + b\bar{\psi}$. We see that φ describes a plane wave with amplitude a coming in from ∞, an amplitude b plane wave being reflected out to ∞ and a unit amplitude wave transmitted to $-\infty$. The corresponding reflection and transmission amplitudes are $r(k) = b/a$ and $t(k) = 1/a$. Constancy of the Wronskian $W = \psi^*\psi_x - \psi_x^*\psi$ (or constancy of the quantum mechanical probability current) then implies that T is unimodular

$$|r|^2 + |t|^2 = 1 \quad \text{or} \quad |a|^2 - |b|^2 = 1 \quad \text{or} \quad \det T = 1. \tag{287}$$

Reflection symmetry: Suppose the Jost function $\varphi(x,k)$ is an eigenfunction of L with eigenvalue k^2, then $\varphi(x,-k)$ is also an eigenfunction with the same eigenvalue. Moreover, since $\varphi(x,k) \to e^{-ikx}$ as $x \to -\infty$, $\phi(x,-k) \to e^{ikx}$ in that limit, just as the eigenfunction $\bar{\varphi}(x,k)$ does. Thus,

$$\varphi(x,-k) = \bar{\varphi}(x,k) \quad \text{and} \quad \psi(x,-k) = \bar{\psi}(x,k). \tag{288}$$

We may use this to show that

$$a(-k) = \bar{a}(k), b(-k) = \bar{b}(k) \quad \text{and consequently} \quad r(-k) = \bar{r}(k). \tag{289}$$

To see this, we use the above result to note that

$$\varphi(x,k) = a(k)\psi(x,k) + b(x,k)\bar{\psi}(x,k) \quad \Rightarrow \quad \varphi(x,-k) = a(-k)\bar{\psi}(x,k) + b(-k)\psi(x,k). \tag{290}$$

Comparing with $\bar{\varphi}(x,k) = \bar{a}(k)\bar{\psi}(x,k) + \bar{b}(k)\psi(x,k)$ we obtain the advertised result. In particular, this means that the reflection amplitude on the positive real axis determines its values on the negative real axis. We will now show that $r(k)$ also determines $a(k)$ and $b(k)$.

[30]The overbar $\bar{\varphi}_1(x,k)$ means we take the conjugate of the function φ_1 of the two variables and evaluate it at x, k. In particular we do not take the complex conjugate of k.

B.2 Minimal scattering data: $a(k)$ from reflection amplitude via dispersion relation

We will assume that $r(k) = b(k)/a(k)$, originally defined for real k extends to a meromorphic function in the complex k plane. This is the case for instance with the point-like attractive potential $V = -g\delta(x)$ where $r(k) = -g/(g + 2ik)$. Moreover, from the asymptotic behavior (116) we see that $a(k)$ must have zeros (which can be shown to be simple) on the positive imaginary axis at $k = i\kappa_n$ for $n = 1, 2, \cdots, N$ so that the bound state wavefunctions corresponding to the discrete spectrum $\lambda_n = -\kappa_n^2$ are normalizable. Consequently, $r(k)$ has simple poles at $k = i\kappa_n$. Now we will indicate how $a(k)$ can be obtained from the values of $r(k)$ on the positive real axis and the locations $i\kappa_n$ of its zeros on the positive imaginary axis.

• The magnitude $|a(k)|$ is easily obtained

$$|t|^2 + |r|^2 = 1 \quad \text{or} \quad \frac{1}{|a|^2} + |r|^2 = 1 \quad \Rightarrow \quad |a| = (1 - |r|^2)^{-1/2}. \tag{291}$$

• Using analyticity, $\arg a(k)$ can be expressed in terms of $|a(k)|$ and the zeros $k = i\kappa_n$ of $a(k)$. Indeed, let us define a new function a_1 with the same modulus as $a(k)$ but which is non-vanishing in the upper half plane, so that its logarithm defines an analytic function

$$a_1(k) = a(k) \prod_1^N \frac{k + i\kappa_n}{k - i\kappa_n}. \tag{292}$$

Each factor in the product is unimodular, so

$$\begin{aligned} \log a_1 &= \log|a_1| + i \arg a_1(k) = \log|a(k)| + i \arg a(k) + \sum_1^N i \arg\left(\frac{k + i\kappa_n}{k - i\kappa_n}\right) \\ \text{or} \quad i \arg a(k) &= i \arg a_1(k) - \sum_1^N \log\left(\frac{k + i\kappa_n}{k - i\kappa_n}\right). \end{aligned} \tag{293}$$

It remains to express $\arg a_1$ in terms of $|a(k)| = |a_1(k)|$. To do this we recall that the Hilbert transform relates the real and imaginary parts of an analytic function. Applying this to $\log a_1(k) = \log|a(k)| + i \arg a_1(k)$ we have

$$\arg a_1(k) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\log|a(k')| dk'}{k' - k}. \tag{294}$$

Putting these together, we get

$$\arg a(k) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\log|a(k')| dk'}{k' - k} + i \sum_1^N \log\left(\frac{k + i\kappa_n}{k - i\kappa_n}\right) \quad \text{where} \quad |a(k)| = \left(1 - |r(k)|^2\right)^{-1/2}. \tag{295}$$

Thus we have expressed both $|a(k)|$ and $\arg a(k)$ (and hence also $a(k)$ and $b(k) = a(k)r(k)$) in terms of $r(k)$ and the κ_n.

Thus, we will regard $r(k)$, for k positive and $\kappa_1, \cdots, \kappa_N$ as part of the scattering data. The last ingredient in the scattering data has to do with the asymptotic behavior of bound state wave functions. Recall that the discrete spectrum $\lambda_n = -\kappa_n^2$ is non-degenerate with wavefunctions

$$\phi_n(x) \to \begin{cases} e^{\kappa_n x} & \text{as} \quad x \to -\infty \\ b_n e^{-\kappa_n x} & \text{as} \quad x \to \infty. \end{cases} \tag{296}$$

Here $b_n = b(i\kappa_n)$ are real by a choice of phase. Moreover, the b_n alternate in sign since ϕ_n has $n-1$ nodes, i.e., $b_1 > 0, b_2 < 0, \ldots$ or $b_n = (-1)^{n-1}|b_n|$. Thus, we will define the scattering data to be the set

$$\mathcal{S} = \{r(k)_{k>0}, \kappa_1, \cdots, \kappa_n, |b_1|, \cdots, |b_n|\} \tag{297}$$

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