Notes on Elasticity, CMI Spring 2018

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These lecture notes are very sketchy and are no substitute for books, attendance and taking notes at lectures. More information will be given on the course web site http://www.cmi.ac.in/~govind/teaching/cont-mech-e18. Please let me know (via govind@cmi.ac.in) of any comments or corrections. Help from Sonakshi Sachdev in preparing these notes is gratefully acknowledged.

Contents

1 Elastostatics

• A wooden ruler, a concrete building/bridge, a stretched string, a steel bar or wooden beam behave as elastic materials when subjected to *any* sufficiently small stress ('stress' = force/area). We say that an object behaves elastically if it tends to retain its original shape and size when any such applied force is withdrawn. By contrast, a fluid is permanently deformed under a small, say shearing, force (it 'flows'). Elasticity is a property both of the material and the applied forces. The same material (e.g. the Earth's crust) may behave elastically for certain small stresses (seismic waves in an earthquake) but suffer non-elastic deformations (rupture/fracture to form a crack/rift (e.g. the great rift valley in East Africa)) when subject to other stresses.

A material suffers plastic deformation if the change in shape is permanent, not reversed when applied stresses are removed, e.g. a thin metal spoon that is beaten or bent into a new shape.

• Elastic deformations are intermediate in complexity between vibrations in a stretched string and fluid flows. Unlike a rigid body, the parts of an elastic body are not rigidly interconnected but suffer small relative displacements due to applied stresses. As in a stretched string, a certain 'ordering' (relative arrangement) of material elements (molecules) is maintained in elastic deformations of a steel beam. But unlike transverse vibrations of a string, elastic seismic waves can have both transverse and longitudinal components associated to 'shearing' and 'compressional' motions. Fluid flows as in air or water can in general mix up the molecules in a much more complicated way.

A material behaves elastically if it retains its shape and size after applied stresses are removed. If the applied force is small enough, the displacements of material elements is proportional to the applied stress. This linear regime continues to hold till the proportionality limit, beyond which the material is still elastic but the resulting strains are not proportional the applied stresses. Many materials (e.g. cemented tungsten carbide) cease to behave elastically for strains greater than 10^{-3}. Beyond the elastic limit, plastic flow sets in and the body does not return to its original shape and size when the stress is removed. At still larger stresses (from the yield point), a small increase in stress can produce a very large strain (it is as if $E \to \infty$). At even larger stresses we reach the rupture point at which the material cracks or breaks.

1.1 Hooke's law of elastic behavior for a cuboid

• We roughly follow Feynman's treatment in Vol 2, Chapt 38 of his Lectures on Physics.

• Consider a cuboid shaped bar of wood/steel with length l, width w and height h. If a tensile (tending to elongate) force F is applied lengthwise at either end of the bar[1], then the increase in length Δl is found to be proportional to the applied force, $\Delta l \propto F$. What is more, if the force is withdrawn, the bar feels a restoring compressional force of the same magnitude F, which is proportional to the elongation $F \propto \Delta l$, with the same constant of proportionality. This is Hooke's law, it can be viewed either way: force producing elongation or elongation producing restoring force. Further, the elongation Δl is proportional to the length of the bar, if we attached two such bars lengthwise, and applied the same force at the ends, the elongation would double. So $\Delta l \propto lF$. In addition, the force needed to produce a given elongation is proportional to the cross sectional area A of the bar. Indeed, if we had two such bars side by side, then we would need to apply twice the force to produce the same elongation in both bars. So Hooke's law becomes

$$F = EA\frac{\Delta l}{l}. \tag{1}$$

where the constant of proportionality E is called Young's modulus of elasticity (some authors denote it by Y, though we will use Y for the tensor of elasticity to be introduced later). E has dimensions of force per unit area, or pressure. It is the analogue of a spring constant. This normal force per unit area F/A is called a (normal) stress. And elongation per unit length $\Delta l/l$ is called strain. So Hooke's law says that stress equals Young's modulus times strain.

[1]If unequal forces are applied on opposite faces, the unbalanced force will accelerate the bar as a rigid body, leaving the balanced part of the forces to cause elastic deformation.

• We will deal primarily with homogeneous and isotropic elastic materials whose properties do not change with location in the material nor with direction. A crystalline material is anisotropic, the same stresses in different directions may produce different strains. For such materials, we need more than one modulus of elasticity to express Hooke's law.

• As one might expect, when a bar is stretched lengthwise by a tensile stress, its dimensions in transverse directions contract. The transverse contractions are proportional to the lengthwise strain and also to the original transverse dimensions. Thus the fractional increase in width and height are negative and equal

$$\frac{\Delta w}{w} = \frac{\Delta h}{h} = -\nu \frac{\Delta l}{l}. \tag{2}$$

In other words, the 'secondary strains' $\Delta w/w$ and $\Delta h/h$ are proportional to the 'primary strain' $\Delta l/l$ with proportionality constant ν. Poisson's ratio ν (sometimes denoted σ) is a dimensionless constant within the limits where Hooke's law holds. It is found to be positive and we will see that $\nu \le \frac{1}{2}$, to ensure that an elastic material does not increase in volume when compressed. It turns out that E and ν are independent material constants that completely specify the elastic properties of homogeneous isotropic materials. In particular, we will see that the bulk modulus (or compressibility) and shear modulus which are other elastic constants can be expressed in terms of E and ν.

• Within the range of validity of Hooke's and Poisson's laws, elastic forces satisfy the superposition principle. The displacements produced by the sum of two forces is the vector sum of the individual displacements.

• Elastic forces, as a consequence of Hooke's law, can have two qualitatively distinct and striking effects on a body, compression and shearing, which we introduce now through examples.

1.2 Hydrostatic compression: bulk modulus

• Suppose our cuboid is placed in water. Assume the hydrostatic pressure p due to the collision of water molecules on all faces is equal (so we ignore the increase in pressure with depth due to gravity). Since hydrostatic pressure is the normal force exerted per unit area, the stress on all faces is equal to p. Let us find the changes in the dimensions l, w, h of the bar and in the volume of the bar.

• By superposition, the lengthwise strain $\Delta l/l$ is the sum of three contributions, contraction $-p/E$ due to lengthwise stress (stresses on the two faces perpendicular to the lengthwise axis), elongation $\nu p/E$ due to breadthwise stresses and elongation $\nu p/E$ due to vertical stresses. Thus $\Delta l/l = -(p/E)(1-2\nu)$. By symmetry, we have

$$\frac{\Delta l}{l} = \frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{p}{E}(1-2\nu) \tag{3}$$

The volume strain is

$$\frac{\Delta V}{V} = \frac{\Delta(lwh)}{lwh} = \frac{\Delta l}{l} + \frac{\Delta w}{w} + \frac{\Delta h}{h} = -\frac{3p}{E}(1-2\nu) \tag{4}$$

The 'volume stress' is just the pressure p, and this formula says it is $\propto$ to the volume strain

$$p = -\frac{E}{3(1-2\nu)} \frac{\Delta V}{V} \equiv -K \frac{\Delta V}{V}. \tag{5}$$

The constant of proportionality $K = \frac{E}{3(1-2\nu)}$ is called the bulk modulus, by analogy with the elastic modulus appearing in Hooke's law. The reciprocal of the bulk modulus is called compressibility $\kappa = 1/K$, which in general is defined as the fractional decrease in volume due to an increase in pressure. In our case, the increase in pressure is p:

$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial p} = \frac{3(1-2\nu)}{E} \tag{6}$$

When the Poisson ratio $\nu = \frac{1}{2}$, we see that the compressibility vanishes, the material is incompressible, it resists a change in volume in the face of compressional stresses. If $\nu > \frac{1}{2}$ the compressibility would be negative, indicating an instability, the block would keep *expanding* under air pressure! Such spontaneous expansion is not observed in ordinary materials, so $\nu < \frac{1}{2}$. However, observations indicate that the universe on a large scale may behave like an unusual material that expands 'by itself' (it seems to have a negative pressure associated to the cosmological constant or conjectured 'dark energy').

1.3 Shearing of a block: shear modulus

• So far we only considered forces that act orthogonal to faces of the block. It is also interesting to consider shearing stresses, forces that act tangential to the faces. We will formulate Hooke's law for general stresses and strains in a later section. For now, let us examine the effect of shearing stresses on a cube by converting it to a problem of normal stresses on a different body, using a clever argument given in Feynman's lectures.

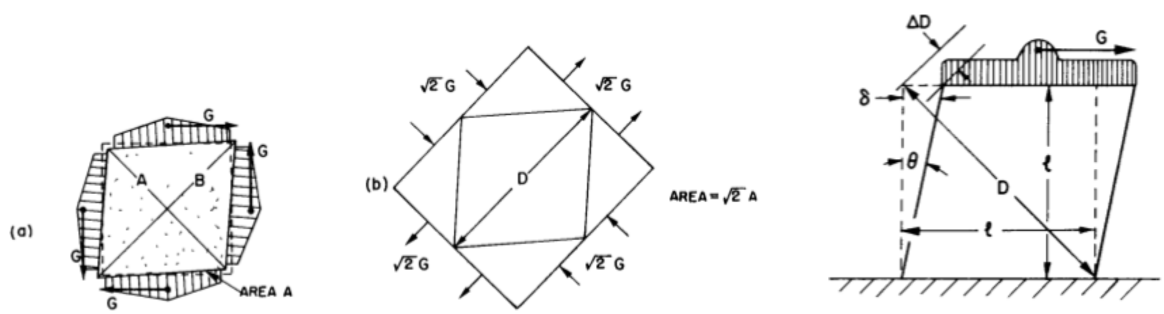

Fig. 38–6. The two pairs of shear forces in (a) produce the same stress as the compressing and stretching forces of (b).

Fig. 38–7. The shear strain θ is $2\,\Delta D/D$.

Figure 1: Shear forces on a cube, scanned from Feynman Lectures Vol 2, Chapter 38.

• Consider a cube of side l viewed normal to a face. It is subjected to 'pure shear' forces of magnitude G acting tangential to the faces of area l^2 as shown in Fig 38-6 of Feynman lectures Vol 2. The shear forces act to the right on the top face and to the left on the bottom face, upwards on the right face and downwards on the left face. They are balanced so that the cube as a whole does not feel any force or torque. The effect of the shearing forces is to deform the cube so that its visible face goes from being a square of diagonal $D = l\sqrt{2}$ to a rhombus with smaller diagonal $D - \Delta D$. We aim to find the strains produced by these stresses. Let us focus on the force acting on two diagonal faces (of depth l into the plane of the paper) obtained by slicing through the cube: **A** (North West-SE) and **B** (SW-NE). The above shearing stresses act normal to these diagonals: (1) stretching/tensile forces of magnitude $\sqrt{2}G$ (the resultant of G on top and G on right) across the NW-SE diagonal face $\mathbf{A}$, acting over an area $\sqrt{2}l^2$ and (2) compressional forces of size $\sqrt{2}G$ acting normal to the SW-NE diagonal face $\mathbf{B}$, again of area $\sqrt{2}l^2$. The pure shear stresses G and strains on the original cube will be obtained from the normal stresses $\sqrt{2}G$ and resulting strains on a hypothetical larger cuboid with sides $l\sqrt{2} \times l\sqrt{2} \times l$, shown in Fig 38-6 (b) (the depth l of the fictitious cuboid is the same as that of the original one). Since we have not yet formulated Hooke's law for non-normal stresses, the correctness of this approach could then be determined by comparing the answers obtained with experiment.

• Let us determine the strains due to compressional and tensile normal stresses of size $G\sqrt{2}$ acting on the faces of area $\sqrt{2}l^2$ of the new cuboid with visible square cross section of side $D = l\sqrt{2}$. One side of the new cuboid (say the one parallel to face $\mathbf{B}$) is elongated by

$$\frac{\Delta D}{D} = \frac{1}{E}\frac{\sqrt{2}G}{\sqrt{2}l^2} + \nu\frac{1}{E}\frac{\sqrt{2}G}{\sqrt{2}l^2} = \frac{(1+\nu)}{E}\frac{G}{l^2} \equiv \frac{(1+\nu)}{E}g. \tag{7}$$

The first term is due to the tensile stress acting normal to the face $\mathbf{A}$ while the second is due to the compressional stress acting on the face B orthogonal to it. The other side is contracted by the same amount. Thus, the result of the pure shear stresses on the original cube is to turn its visible cross section into a rhombus with diagonals $D \pm \Delta D$.

• To identify the shear modulus, we first note that $g = G\sqrt{2}/\sqrt{2}l^2 = G/l^2$ is the shear stress. It is tempting to define $\Delta D/D$ as the shear strain, but it it conventional to define the shear strain as the angle θ by which the cube is sheared, so that its cross section looks like a rhombus. $\theta = \delta/l$ where δ is the tangential displacement of any face. Fig 38-7 shows that $\delta = \sqrt{2}\Delta D$ (consider the small isosceles right triangle with sides $\Delta D, \Delta D$ and δ) where $D = \sqrt{2}l$. So the shear stress $g = G/l^2$ may be expressed as a multiple of the shear strain

$$g = \frac{E}{1+\nu}\frac{\Delta D}{D} = \frac{E}{1+\nu}\frac{\delta}{\sqrt{2}\times\sqrt{2}l} = \frac{E}{2(1+\nu)}\frac{\delta}{l} \equiv \mu\theta \quad \text{where} \quad \mu = \frac{E}{2(1+\nu)}. \tag{8}$$

μ is called the shear modulus or coefficient of rigidity or Lamé's first elastic constant. Like Young's modulus, it must be positive, since elastic forces are *restoring* forces.

1.4 Stress tensor

• More generally, forces need not act either normal or tangential to surfaces[2] of the material, and they could vary in magnitude and direction with location in the material. A device that encodes the force per unit area acting across an element of surface is called the stress tensor. Let $\mathbf{n}\,\delta A$ be a small surface element of area δA, with unit normal $\mathbf{n}$, centered at $\mathbf{x}$. Let $\mathbf{F}(\mathbf{n}\delta\mathbf{A}, \mathbf{x})$ be the force that acts across the surface. (Feynman works with the force per unit area $\Sigma(\hat{\mathbf{n}}, \mathbf{x}) = \mathbf{F}/\delta A$) Precisely, it is the force on the material on the side to which $\mathbf{n}$ points, due to the material on the other side.

• In general $\mathbf{F}$ and $\mathbf{n}$ point in different directions and are related by a linear transformation, the transformation of stress. If we choose to write all vectors in some basis, e.g. resolve them according to cartesian components, then this linear relation may be written as

$$\mathbf{F}_i(\mathbf{n}\delta A, \mathbf{x}) = \sum_j T_{ij}(\mathbf{x}) n_j \,\delta A. \tag{9}$$

[2]These surfaces may be external surfaces or, more frequently, hypothetical internal surfaces in the material.

The 3×3 matrix $T_{ij}(\mathbf{x})$ is called the stress tensor field. It depends only on the location $\mathbf{x}$ and not on the surface or $\mathbf{n}$. By choosing a surface whose normal $\mathbf{n}$ points in the j direction, we see that T_{ij} is then the i^{th} component of the force acting on the material towards the j^{th} direction of a surface of unit area whose normal points in the j^{th} direction. Alternately, suppose δA is a small surface with normal $\hat{n}$, then $\sum_j T_{ij} n_j (\delta A)$ is the i^{th} component of the force acting on the material on the side to which the normal $\hat{n}$ points.

Remark: Consider a small sphere enclosing a volume of material with outward normal $\hat{n}$. If $\mathbf{F}$ points in the same direction as $\mathbf{n}$, then it is a tensile stress, tending to pull the surface in the 'outward' direction and thereby expand the sphere. If $\mathbf{F}$ points anti-parallel to $\mathbf{n}$, then it is a compressional stress tending to compress the sphere.

• **Stress tensor due to hydrostatic pressure** at a point in water. By definition, hydrostatic pressure acts normal to any surface considered. So consider a small cuboid with axes along the cartesian axes, it follows that $T_{ij} = 0$ for $i \neq j$ since there are no tangential stresses. Moreover, $T_{33} = p$ since the force across the top surface, whose normal points in $\hat{z}$ direction due to the fluid below is $p\hat{z}$. We get the same answer $T_{33} = p$ by considering the bottom surface. Proceeding in this way $T_{ij} = p\delta_{ij}$ due to hydrostatic pressure. This formula for the stress tensor due to hydrostatic pressure is independent of basis, multiples of the identity matrix have the same components in any basis.

• **Stress tensor for tensile elongation of a rectangular bar:** Consider the tensile elongation of a bar of length l and cross sectional area A by the application of a tensile stress F/A in the $\hat{x}$ direction on the right face and F/A in the $-\hat{x}$ direction on the left face. So there is a stress F/A in the same direction as the normal $\hat{x}$ to the right face, due to the external agent who is on the $+x$ side of the face. So $T_{11} = -F/A$. We also obtain $T_{11} = -F/A$ by considering the left face. There is no force in the y, z directions anywhere. So T_{2j} and T_{3j} are all zero. There is no force on the remaining faces, so $T_{i2} = 0$ and $T_{i3} = 0$.

$$\text{So the stress tensor is} \quad T_{ij} = -\frac{F}{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

This stress tensor is clearly anisotropic unlike that for hydrostatic pressure. The difference in sign is due to the tensile force on the bar, unlike the compressional force of hydrostatic pressure.

• **'Pure shear' stress tensor:** Suppose we have a material subject to pure shear forces g per unit area, acting on any small cube. More precisely, an external agent applies a force g per unit area on the top, bottom, right and left faces of an elementary cube, in the $x, -x, y, -y$ directions, as in Feynman's example considered in the previous section. Let us find the stress tensor. Since there is no force in the z direction, $T_{3j} = 0$ for $j = 1, 2, 3$. Further, the diagonal components $T_{ii} = 0$ since there are no normal stresses. To find T_{12}, consider the top surface. T_{12} is the x component of the stress on the top surface due to the material inside the cube, this is a restoring force and is just $-g$. We get the same answer by considering the bottom surface: T_{12} is the x-component of stress on the bottom surface due to material outside the cube and it is equal to $-g\hat{x}$. It is the negative of the x-component of the stress on the bottom surface due to material inside the cube. So we get $T_{12} = -g$ by considering either the top or bottom face. By considering the right face we find T_{21} is the y-component of the stress on the right face due to material inside the cube. Thus $T_{21} = -g$. Thus we find

$$\mathbf{T} = -g \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

To say that this stress tensor is 'pure shear' or purely off-diagonal, is a basis-dependent statement. As Feynman did in the above example, we may write the stress tensor in a different basis, by $\pi/4$ rotation. In the rotated basis it is diagonal and looks like a combination of normal tension and compression in two orthogonal eigen-directions $\tilde{\mathbf{T}} = -g \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

1.4.1 Symmetry of the stress tensor

• In the above examples, we notice that the stress tensor is a symmetric matrix at each point. We argue now, that this must always be true. Consider a volume V of material surrounded by a surface $A = \partial V$. The $i^{\rm th}$ component of the force per unit area acting on the material inside across any point of this surface is $-T_{ij} n_j$ where $\hat{n}$ is the unit outward normal. Let us consider the rate of change of angular momentum of the fluid in V about an origin located inside V. It must equal the torque due to surface forces acting on the fluid inside V plus those due to body forces ($\mathbf{f}^{\rm b}$ per unit volume, e.g. $\rho\vec{g}$ in the case of gravity where $\vec{g}$ is the acceleration due to gravity)

$$\begin{aligned} \frac{d}{dt}\int_V (\mathbf{r}\times\mathbf{v})_i \,\rho\, dV &= \int (\mathbf{r}\times\mathbf{f}^{\rm b})_i dV - \int_A \epsilon_{ijk} r_j T_{kl} n_l dA = \int_V \left[(\mathbf{r}\times\mathbf{f}^{\rm b})_i - \epsilon_{ijk}\partial_l (r_j T_{kl})\right] dV \\ &= \int_V \left[(\mathbf{r}\times\mathbf{f}^{\rm b})_i - \epsilon_{ijk} T_{kj} - \epsilon_{ijk} r_j \partial_l T_{kl}\right] dV \end{aligned} \tag{12}$$

Here ρ is the mass density and $\mathbf{v}$ the velocity of the material at the position $\mathbf{r}$. ϵ_{ijk} is the Levi-civita symbol, it is anti-symmetric under exchange of any pair of indices with $\epsilon_{123} = 1$, so that $\epsilon_{132} = -1$ and $\epsilon_{112} = 0$ for instance, it gives a convenient way of writing the cross product. The second equality is a consequence of Gauss' divergence theorem which states $\int_{\partial V} v_l n_l dA = \int_V \partial_l v_l dV$. Now we let V shrink to zero, thinking of it as a spherical volume so that its linear dimension is of order $V^{1/3}$ and $|\mathbf{r}| \lesssim V^{1/3}$ (here it where it helps to have the origin inside the volume). $\rho, \mathbf{v}$, $\mathbf{f}^{\rm b}$, T_{kj} and $\partial_l T_{kl}$ all have finite limits independent of the volume V, they approach their limiting values at the 'central' location to which the volume shrinks. The lhs scales like $V^{4/3}$ due to the $\mathbf{r}$ and dV factors. The body force term and the last term on the rhs also scale like $V^{4/3}$ due to the factors of $\mathbf{r}$ and dV. On the other hand, the middle term on the rhs scales like V. For small V, V is larger than $V^{4/3}$. For the equation to hold, the un-balanced middle torque term must vanish identically for any i and for a small volume of any shape located anywhere in the fluid. So $\epsilon_{ijk} T_{jk}$ must be zero for all i. When written out, this implies $T_{12} = -T_{21}, T_{13} = -T_{31}$ and $T_{23} = -T_{32}$. So the stress tensor is a symmetric tensor field $T_{ij}(x) = T_{ji}(x)$, it has only 6 independent components at any point. The diagonal elements represent normal stresses while the off diagonal components are tangential (or shearing) stresses.

1.5 Tensor of strain

• Just as the stresses in a material are encoded in a symmetric tensor field, so also with the resulting strains. Let us consider an infinitesimal distortion in a material, with each material point originally located at $\mathbf{r} = (x^1, x^2, x^3) = (x, y, z)$ moving to a nearby point $\mathbf{r} + \xi(\mathbf{r})$. We call $\xi(\mathbf{r})$ the displacement field, it gives the infinitesimal displacement as a function of original location. Now if $\xi(\mathbf{r})$ were independent of location, then all points in the body would be uniformly translated. This is simply the center of mass motion of a rigid body, and is not related to elastic properties. In straining motion, the displacement field must depend non-trivially on $\mathbf{r}$.

• However, if the infinitesimal displacements are rotations $\xi = \mathbf{r} \times \delta\vec{\phi}$ (rotation of $\mathbf{r}$ by angle $\delta\phi$ clockwise about an axis along $\delta\vec{\phi}$), then again, we do not classify it as a strain. On the other hand, a displacement field such as $\xi = (x\Delta l/l, 0, 0)$ describes a strain due to stretching of a bar of length l by Δl in the x-direction (ignoring possible contraction in other directions). It turns out that the symmetric tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i}\right) \tag{13}$$

encodes the local strains. Let us see why. Consider a small region and place the origin inside it. Assuming the displacements are small, we expand the displacement field in a Taylor series around $\mathbf{r} = 0$ (the same could be done around any point)

$$\xi = \xi^0 + (\mathbf{r}\cdot\nabla)\xi + \mathcal{O}(\mathbf{r})^2 \quad \text{or} \quad \xi_i = \xi_i^0 + x^j \frac{\partial \xi_i}{\partial x^j} + \cdots \tag{14}$$

Here and elsewhere, repeated indices are summed from one to three. ξ^0 represents uniform translation of material in the whole region. Here partial derivatives are evaluated at the origin $\mathbf{r} = 0$. By adding and subtracting, we re-write this as

$$\xi = \xi^0 + x^j\left[\frac{(\partial_j\xi_i + \partial_i\xi_j)}{2} + \frac{\partial_j\xi_i - \partial_i\xi_j}{2}\right] + \cdots \;=\; \xi^0 + e_{ij}x^j + \omega_{ij}x^j + \cdots \tag{15}$$

Here we defined the symmetric strain tensor e_{ij} as advertised before, and the anti-symmetric rotation (or vortex/vorticity) tensor ω_{ij}. The components ω_{ij} appear in the curl $\nabla \times \xi$, in fact verify that $(\nabla \times \xi)_i = \epsilon_{ijk}\partial_j\xi_k = -2\epsilon_{ijk}\omega_{jk}$. $\omega_{ij}x^j$ represents an infinitesimal rotation of $\mathbf{r}$ about the vector $\delta\vec{\phi} = (\delta\phi_1, \delta\phi_2, \delta\phi_3) = (\omega_{23}, \omega_{31}, \omega_{12})$

$$\begin{aligned} \omega_{ij}x_j &= (\omega_{12}x_2 + \omega_{13}x_3, \omega_{21}x_1 + \omega_{23}x_3, \omega_{31}x_1 + \omega_{32}x_2) \\ &= (x_2\delta\phi_3 - x_3\delta\phi_2, x_3\delta\phi_1 - x_1\delta\phi_3, x_1\delta\phi_2 - x_2\delta\phi_1) = (\vec{r}\times\delta\vec{\phi})_i \end{aligned} \tag{16}$$

This decomposition of a general displacement field into a translation, rotation and strain was introduced by Helmholtz in the context of fluid mechanics. Proceeding as above in each elemental region of the material, one defines the rotation and strain tensor fields e_{ij}, ω_{ij} all over the material.

• It is also convenient to define the 2nd rank tensor field $S = \nabla\xi$ or $S_{ij} = \partial_j\xi_i$ which may be decomposed into its symmetric and anti-symmetric parts

$$S_{ij} = e_{ij} + \omega_{ij} \quad \text{where} \quad e_{ij} = \frac{1}{2}(S_{ij} + S_{ji}), \;\; \omega = \frac{1}{2}(S_{ij} - S_{ji}). \tag{17}$$

The symmetric part e_{ij} may be further decomposed into its trace and trace-free parts:

$$e_{ij} = \Sigma_{ij} + \frac{1}{3}\delta_{ij} \text{ tr } \mathbf{e} \quad \text{where} \quad \Sigma_{ij} = \frac{1}{2}(\partial_j \xi_i + \partial_i \xi_j) - \frac{1}{3}\delta_{ij} \nabla \cdot \xi. \tag{18}$$

$\Theta = \text{ tr } \mathbf{e} = S_{ii} = \nabla \cdot \xi$ is a scalar field called the expansion. The traceless tensor field Σ is called the shear tensor.

• These formulae are valid in Cartesian coordinates on Euclidean space. In curvilinear coordinates (or more generally, on curved manifolds) we would need to replace partial derivatives with covariant derivatives and replace δ_{ij} with the components of the metric tensor field.

• In an elastic material under given stresses, our aim is often to calculate the resulting displacement field $\xi(\mathbf{r}, t)$, just as we wish to find the displacement $u(x,t)$ (from equilibrium), of a vibrating stretched string. In general, the displacement field may involve rigid translations, (local) rotations (with $\nabla \times \xi \neq 0$), as well as straining motions. But it is only the strain tensor e_{ij} that will appear in the tensorial formulation of Hooke's law, which eventually leads to the equations of elastostatics and elastodynamics.

• **Strain tensor for hydrostatic pressure:** Consider a cuboid with one vertex fixed at the origin and under hydrostatic compression. The length, breadth and height are compressed by $\Delta l, \Delta w, \Delta h$. We know that $\frac{\Delta l}{l} = \frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{p(1-2\nu)}{E}$. Thus, the displacement field is $\xi = \frac{\Delta l}{l}(x, y, z)$. The resulting strain tensor is a multiple of the identity $e_{ij} = \frac{\Delta l}{l}\delta_{ij} = -\frac{p(1-2\nu)}{E}\delta_{ij}$.

• **Strain tensor for a bar under tensile stress:** Consider the elongation of a bar of length l and cross sectional area A by the application of a tensile stress F/A in the $\hat{x}$ direction on the right face. The left end of the bar does not move horizontally with one of its corners held fixed at the origin. The tensile force elongates the bar in the x direction and contracts it in the orthogonal directions. By Hooke's law we found

$$\frac{\Delta l}{l} = \frac{F}{AE}, \quad \text{and} \quad \frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{\nu \Delta l}{l} = -\frac{\nu F}{AE}. \tag{19}$$

Thus the displacement field is

$$\xi = \left(x\frac{\Delta l}{l}, y\frac{\Delta w}{w}, z\frac{\Delta h}{h}\right) = \left(\frac{xF}{AE}, -\frac{y\nu F}{AE}, \frac{-z\nu F}{AE}\right) = \frac{F}{AE}(x, -y\nu, -z\nu) \tag{20}$$

It follows that the strain tensor $e_{ij} = \frac{1}{2}(\partial_i \xi_j + \partial_j \xi_i)$ is diagonal

$$e_{ij} = \frac{F}{AE}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}, \quad \Sigma = \frac{F}{3AE}\begin{pmatrix} 2(1+2\nu) & 0 & 0 \\ 0 & -\nu-1 & 0 \\ 0 & 0 & -\nu-1 \end{pmatrix} \quad \text{and} \quad \Theta = \frac{F}{AE}(1-2\nu). \tag{21}$$

• **'Pure shear' strain tensor:** The strain tensor $e_{ij} = \epsilon \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ represents an infinitesimal shearing strain corresponding to the displacement field $\xi = \epsilon(y, x, 0)$. When plotted we see that this corresponds to a local pure shear of the sort caused by the pure shear stress studied in Feynman's example. Note that this ξ is curl-free $\nabla \times \xi = 0$.

1.5.1 Volume strain, divergence of displacement field and trace of strain tensor

• The volume strain $\Delta V/V$ for a small volume V located near $\mathbf{r}$ is encoded in the trace of the strain tensor $\text{tr}\ e = e_{ii}$, which is the same as the divergence of the displacement field $\nabla \cdot \xi = \partial_i \xi_i = \Theta$. Let us see why. Consider a small cuboid with a vertex at $\mathbf{r}$ and edges along $\delta\vec{x}, \delta\vec{y}, \delta\vec{z}$. The volume of this cuboid is the determinant of a matrix whose rows are these three vectors $\det(\delta\vec{x}, \delta\vec{y}, \delta\vec{z})$. Now suppose the material is deformed with a displacement field $\xi(\mathbf{r})$. It maps the vertex $\mathbf{r} \to \mathbf{r} + \xi$ and the vertex

$$\mathbf{r} + \delta\vec{x} \mapsto \mathbf{r} + \delta\vec{x} + \xi(\mathbf{r} + \delta\vec{x}) = \mathbf{r} + \delta\vec{x} + \xi(\mathbf{r}) + \delta\vec{x} \cdot \nabla\xi \tag{22}$$

So edge vector $\delta\vec{x}$ is transformed into

$$\mathbf{r} + \delta\vec{x} + \xi(\mathbf{r}) + \delta\vec{x} \cdot \nabla\xi - \mathbf{r} - \xi = \delta\vec{x} + \delta\vec{x} \cdot \nabla\xi \tag{23}$$

So the volume of the deformed cube is

$$\text{New vol} \approx \det \begin{pmatrix} \delta\vec{x} + \delta\vec{x} \cdot \nabla\xi \\ \delta\vec{y} + \delta\vec{y} \cdot \nabla\xi \\ \delta\vec{z} + \delta\vec{z} \cdot \nabla\xi \end{pmatrix} = \det \begin{pmatrix} \delta\vec{x} \\ \delta\vec{y} \\ \delta\vec{z} \end{pmatrix} + \det \begin{pmatrix} \delta\vec{x} \cdot \nabla\xi \\ \delta\vec{y} \\ \delta\vec{z} \end{pmatrix} + \det \begin{pmatrix} \delta\vec{x} \\ \delta\vec{y} \cdot \nabla\xi \\ \delta\vec{z} \end{pmatrix} + \det \begin{pmatrix} \delta\vec{x} \\ \delta\vec{y} \\ \delta\vec{z} \cdot \nabla\xi \end{pmatrix} \tag{24}$$

Here we repeatedly used the fact that the determinant is linear in each row and dropped terms quadratic and higher order in small quantities. The first term is the old volume. To find the change in volume per unit volume, let us choose the edges of the original cube to lie along the Cartesian axes: $\delta\vec{x} = \delta x\, \hat{x}, \delta\vec{y} = \delta y\, \hat{y}, \delta\vec{z} = \delta z\, \hat{z}$. Then the old volume is $V = \delta x \delta y \delta z$ and the change in volume is

$$\begin{aligned} \Delta V &= \det \begin{pmatrix} \delta x\, \partial_1 \xi_1 & 0 & 0 \\ 0 & \delta y & 0 \\ 0 & 0 & \delta z \end{pmatrix} + \det \begin{pmatrix} \delta x & 0 & 0 \\ 0 & \delta y\, \partial_2 \xi_2 & 0 \\ 0 & 0 & \delta z \end{pmatrix} + \det \begin{pmatrix} \delta x & 0 & 0 \\ 0 & \delta y & 0 \\ 0 & 0 & \delta z\, \partial_3 \xi_3 \end{pmatrix} \\ &= (\delta x\, \delta y\, \delta z)(\partial_1 \xi_1 + \partial_2 \xi_2 + \partial_3 \xi_3) = V\, \nabla \cdot \xi. \end{aligned} \tag{25}$$

Recalling that $e_{ij} = \frac{1}{2}(\partial_i \xi_j + \partial_j \xi_i)$, we see that the change in volume per unit volume, or volume strain, equals the trace of the strain tensor $\frac{\Delta V}{V} = \nabla \cdot \xi = \ \text{tr}\ e = \Theta$. In particular, if the material is incompressible to elastic deformations, then the strain tensor must be traceless.

• For example, let us find the volume strain due to hydrostatic pressure. Consider a cuboid with sides l, w, h subject to hydrostatic pressure p on all faces. Earlier we used superposition to find

$$\frac{\Delta l}{l} = \frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{p(1-2\nu)}{E} \tag{26}$$

It follows that the volume strain is

$$\frac{\Delta V}{V} = \frac{\Delta l}{l} + \frac{\Delta w}{w} + \frac{\Delta h}{h} = -\frac{3p(1-2\nu)}{E} \tag{27}$$

This agrees with the trace of the strain tensor $e_{ij} = -\frac{p}{E}(1-2\nu)\delta_{ij}$, $\ \text{tr}\ \mathbf{e} = -\frac{3p}{E}(1-2\nu)$.

1.6 Tensorial form of Hooke's law

• Thus in general, both the local stresses and strains in an elastic material are specified by second rank symmetric tensors T_{ij} and e_{ij}. Hooke's law then says that the components of the stress tensor are linear in the components of the strain tensor

$$T_{ij}(\mathbf{r}) = -\sum_{kl} Y_{ijkl}(\mathbf{r}) e_{kl}(\mathbf{r}) \tag{28}$$

The minus sign is conventional and signifies that the stress is a restoring one. The coefficients $Y_{ijkl}(\mathbf{r})$ that relate the two tensors is a fourth rank tensor field called the tensor of elasticity. It is symmetric in the first pair of indices (as T_{ij} is) and may be taken symmetric in the second pair as well (as e_{kl} is symmetric and any part of Y anti-symmetric in the last pair would not contribute). When this is done, we may also write $T_{ij} = -Y_{ijkl}S_{kl}$ as the anti-symmetric rotation part of S does not contribute. This leaves Y_{ijkl} with $6\times 6 = 36$ independent components. Thus we may regard Y as a linear transformation from one space of symmetric 3×3 matrices to another such space, i.e., a map from $\mathbb{R}^6 \to \mathbb{R}^6$. We will see later that Y is also symmetric under the exchange of the 1st and 2nd pairs of indices (in other words, Y is a symmetric operator from $\mathbb{R}^6 \to \mathbb{R}^6$) leaving only 21 independent components at each location $\mathbf{r}$. To see this, we regard ij and kl each as a single index which can take 6 values. A 2nd rank symmetric tensor on a six dimensional space has $6\times 7/2 = 21$ components.

• If the material is homogeneous, then Y_{ijkl} are independent of location. Particular components of Y are related to Young's modulus of elasticity and Poisson's ratio in various directions. When stresses T_{ij} are applied, strains develop so as to satisfy the above equation, and the body acquires a corresponding distorted shape and reaches a new equilibrium. Alternatively, if an elastic body is deformed by the displacement field ξ, then restoring stresses in the amount $T_{ij} = -Y_{ijkl}e_{kl}$ develop in the material.

• For isotropic materials (whose elastic properties are independent of direction at any point), the tensor of elasticity is an isotropic tensor with only two independent components. To understand this we need some definitions and facts about Cartesian tensors.

1.6.1 Isotropic Cartesian Tensors

• Consider the vector space R^3 with an ordered orthonormal basis. By rotations we may obtain other orthonormal bases with the same orientation. These rotations form the special orthogonal group SO(3) consisting of 3×3 matrices R satisfying $R^t R = I$ and $\det R = 1$. An orthogonal matrix with $\det R = -1$ would reverse the orientation of the basis. A scalar is a real-valued quantity that has the same value in all bases. A tensor of rank one (or a vector) under rotations is a triple (v_1, v_2, v_3) whose components transform to $v_i' = \sum_j R_{ij} v_j$ under rotations. A tensor of rank two is a set of nine quantities t_{ij} whose values in different o.n. bases are related by $t'_{ij} = \sum_{kl} R_{ik} R_{jl} t_{kl}$. A simple example of such a tensor is the outer or tensor product of two vectors v and w: $t_{ij} = v_i w_j$. More generally we define Cartesian tensors of rank n as a set of 3^n real quantities $t_{i_1\cdots i_n}$ whose values in different o.n. bases are related by $t'_{i_1\cdots i_n} = \sum_{j_1\cdots j_n} R_{i_1 j_1}\cdots R_{i_n j_n} t_{j_1\cdots j_n}$. A simple way fo constructing such tensors is by taking the tensor product of tensors of lower rank: the tensor product of tensors of ranks p and q gives us a tensor of rank $p+q$: $t_{i_1\cdots i_{p+q}} = r_{i_1\cdots i_p} s_{i_{p+1}\cdots i_{p+q}}$.

A Cartesian tensor is said to be isotropic or rotation-inavariant if its components have the same numerical values in all o.n. coordinate frames. In other words,

$$t_{i_1\cdots i_n} = \sum_{j_1\cdots j_n} R_{i_1 j_1}\cdots R_{i_n j_n} t_{j_1\cdots j_n} \tag{29}$$

for all $i_1\cdots i_n$ and all rotations R. A scalar is, by definition, an isotropic tensor. The only isotropic tensor of rank one is the zero vector. After all, the components of a non-zero vector *do change* under a rotation about an axis that does not point along the vector. The only isotropic tensors of rank two are scalar multiples of the identity matrix: δ_{ij}. The only isotropic tensors or rank three are scalar multiples of the Levi-Civita tensor ϵ_{ijk}. To show this, notice that

$$\epsilon'_{123} = R_{1i}R_{2j}R_{3k}\epsilon_{ijk} = \det R = 1 \tag{30}$$

and similarly $\epsilon'_{132} = -\det R = -1$ etc.

• A simple way of obtaining isotropic tensors of higher rank is by taking the tensor products of isotropic tensors of lower rank. In particular, products of Kronecker deltas and Levi-Civita tensors may be used to construct interesting isotropic tensors. In fact, it may be shown that every such tensor is a linear combination of such products.

• The space of isotropic tensors of rank four is three dimensional. Any such tensor is a linear combination of a product of two Kronecker deltas with all possible choices of indices:

$$Y_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_1\delta_{ik}\delta_{jl} + \mu_2\delta_{il}\delta_{jk}. \tag{31}$$

1.6.2 Tensor of elasticity for an isotropic material and Lamé's constants

• For an isotropic material, Y_{ijkl} must be an isotropic tensor of rank four that is symmetric in the first pair and last pair of indices and also under the exchange of the firs pair with the last pair. These conditions require that $\mu_1 = \mu_2 \equiv \mu$ above, so that

$$Y_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}). \tag{32}$$

Hooke's law now reads

$$T_{ij} = -Y_{ijkl}e_{kl} = -\lambda\delta_{ij}e_{kk} - 2\mu e_{ij} = -\lambda\delta_{ij}\nabla\cdot\xi - \mu(\partial_i\xi_j + \partial_j\xi_i) = -\left(\lambda + \frac{2}{3}\mu\right)\delta_{ij}\nabla\cdot\xi - 2\mu\Sigma_{ij}. \tag{33}$$

Since e_{kl} is dimensionless, Y_{ijkl} and μ, λ have dimensions of stress $ML^{-1}T^{-2}$. μ and λ are the first and second of Lamé's elastic constants.

On physical grounds, we might expect the coefficient of the expansion term to be proportional to the bulk modulus K and and that of the shear tensor Σ_{ij} to be proportional to the shear modulus. It turns out that μ is in fact the shear modulus introduced earlier and that

$$T_{ij} = -K\Theta\delta_{ij} - 2\mu\Sigma_{ij}. \tag{34}$$

We may show this by considering an example which allows us to express μ and λ in terms of Young's modulus E and Poisson's ratio ν which were in turn related to the bulk and shear moduli.

1.6.3 Relation between Lamé's constants and Young's modulus and Poisson ratio

• Let us try to relate λ, μ to E, ν. Consider the stress and strain due to hydrostatic pressure on a cuboid. We found

$$T_{ij} = p\delta_{ij} \quad \text{and} \quad e_{ij} = -\frac{p(1-2\nu)}{E}\delta_{ij} \quad \Rightarrow \quad \text{tr}\, e = -\frac{3p(1-2\nu)}{E} \tag{35}$$

Inserting in the above stress vs strain relation $T_{ij} = -\lambda\delta_{ij}\,\text{tr}\, e - 2\mu e_{ij}$ we get

$$p\delta_{ij} = \lambda\delta_{ij}\frac{3p(1-2\nu)}{E} + 2\mu\frac{p(1-2\nu)}{E}\delta_{ij} \quad \Rightarrow \quad E = (3\lambda + 2\mu)(1-2\nu) \tag{36}$$

Due to the isotropy of hydrostatic pressure, we get only one relation between λ, μ and E, ν. To get another relation we must consider a situation where stress and strain aren't both proportional to the identity.

• **Relation between λ, μ and E, ν via elongation of a rectangular bar:** Let us relate λ, μ to E, ν by looking at the example of elongation of a bar of length l and cross sectional area $w \times h$ by the application of a tensile stress g in the $\hat{x}$ direction on the right face. The left end of the bar is held fixed, in fact, a corner of the bar that is located at the origin is held fixed. The tensile force elongates the bar in the x direction and contracts it in the orthogonal directions. By Hooke's law we found

$$\frac{\Delta l}{l} = \frac{g}{E}, \quad \text{and} \quad \frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{\nu\Delta l}{l} = -\frac{\nu g}{E}. \tag{37}$$

Thus the displacement field is

$$\xi = \left(x\frac{\Delta l}{l}, y\frac{\Delta w}{w}, z\frac{\Delta h}{h}\right) = \left(\frac{xg}{E}, -\frac{y\nu g}{E}, \frac{-z\nu g}{E}\right) = \frac{g}{E}\left(x, -y\nu, -z\nu\right) \tag{38}$$

It follows that the strain tensor $e_{ij} = \frac{1}{2}(\partial_i\xi_j + \partial_j\xi_i)$ is diagonal

$$e_{ij} = \frac{g}{E}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix} \tag{39}$$

What about the stress tensor? The only stress is g in the $\hat{x}$ direction by the right face. So T_{2j} and T_{3j} are all zero. By symmetry, T_{11} is the only non-vanishing component. So, the stress tensor is

$$T_{ij} = -g\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

Now Hooke's law $T_{ij} = -\lambda\delta_{ij}e_{kk} - 2\mu e_{ij}$ says

$$-g\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{g}{E}\begin{pmatrix} \lambda(1-2\nu) + 2\mu & 0 & 0 \\ 0 & \lambda(1-2\nu) - 2\mu\nu & 0 \\ 0 & 0 & \lambda(1-2\nu) - 2\mu\nu \end{pmatrix} \tag{41}$$

Comparing diagonal entries, we read off two relations

$$E = 2\mu + \lambda(1-2\nu) \quad \text{and} \quad 0 = \frac{g}{E}\left(\lambda(1-2\nu) - 2\mu\nu\right) \tag{42}$$

We use these to express E, ν in terms of Lamé's constants

$$E = \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \tag{43}$$

Or conversely, we may express Lame's constants in terms of E, ν

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \tag{44}$$

We argued that $E \geq 0$ and $0 \leq \nu \leq \frac{1}{2}$ so $\lambda \geq 0$ and $\mu \geq 0$.

• In some treatments, one works with the bulk modulus K and the shear modulus μ instead of the pair (λ, μ) or (E, ν). Recall that $K = \frac{E}{3(1-2\nu)}$ so that

$$K = \lambda + \frac{2}{3}\mu. \tag{45}$$

1.6.4 Eigenvalue problem for Y_{ijkl} for an isotropic material

• The tensor of elasticity Y_{ijkl} is a linear operator on the space of symmetric second rank (strain) tensors: $e_{kl} \mapsto Y_{ijkl} e_{kl}$. Since the space of strain tensors is six dimensional, Y may be regarded as a 6×6 matrix once we choose a suitable basis for symmetric tensors. It must have six eigenvalues. Moreover, it is a symmetric operator as $Y_{ijkl} = Y_{klij}$. Thus its eigenvalues must be real and it must be diagonalizable. We wish to find its eigenvalues and eigenvectors in the case when the material is isotropic, so that $Y_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

• The only isotropic matrices are multiples of the identity. So it is natural to check if the Kronecker delta is an eigenvector of Y:

$$Y_{ijkl}\delta_{kl} = \lambda\delta_{ij}\delta_{kl}\delta_{kl} + \mu(\delta_{ik}\delta_{jl}\delta_{kl} + \delta_{il}\delta_{jk}\delta_{kl}) = (3\lambda + 2\mu)\delta_{ij}, \tag{46}$$

using $\delta_{kk} = 3$. Thus scalar strain tensors $e_{kl} = \frac{1}{3}\Theta\delta_{kl}$ are eigenvectors of Y with eigenvalue $3\lambda + 2\mu = 3K$. Physically, this is very reasonable: it says that every pure expansion is an eigenvector of Y with eigenvalue given by thrice the bulk modulus.

To get some insight into the remaining eigenvalues, let us compute the trace of Y:

$$\operatorname{tr} Y = Y_{ijij} = \lambda\delta_{ij}\delta_{ij} + \mu(\delta_{ii}\delta_{jj} + \delta_{ij}\delta_{ji}) = 3\lambda + 12\mu. \tag{47}$$

Thus the sum of the remaining five eigenvalues must be 10μ. We might even guess that all five of these are equal to 2μ. But how will we find out?

• Since there are no more isotropic second rank tensors, we wonder what the remaining eigenvectors may be. To find them it will be convenient to choose a basis for symmetric 3×3 strain tensors consisting of the following six 'elementary' matrices. Three span the diagonal subspace

$$w_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{48}$$

and the remaining three span the space of off diagonal symmetric tensors

$$w_{12}+w_{21}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad w_{23}+w_{32}=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad w_{31}+w_{13}=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}. \tag{49}$$

In particular $\delta = w_{11}+w_{22}+w_{33}$. Here the matrix elements of w_{ij} are

$$(w_{ij})_{kl} = \delta_{ik}\delta_{jl}. \tag{50}$$

Now we find the images of these basis vectors under the application of Y. Begin with w_{11}:

$$(Yw_{11})_{ij} = Y_{ijkl}(w_{11})_{kl} = (\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}))\delta_{k1}\delta_{l1} = \lambda\delta_{ij}+2\mu\delta_{1i}\delta_{j1} \tag{51}$$

Thus we get

$$\begin{aligned} Yw_{11} &= (\lambda+2\mu)w_{11}+\lambda(w_{22}+w_{33}), \quad Yw_{22}=(\lambda+2\mu)w_{22}+\lambda(w_{11}+w_{33}),\\ \text{and}\quad Yw_{33} &= (\lambda+2\mu)w_{33}+\lambda(w_{11}+w_{22}). \end{aligned} \tag{52}$$

In other words, the diagonal matrices w_{11}, w_{22}, w_{33} span an invariant subspace though none of them is an eigenvector. However, it follows that traceless diagonal matrices are eigenvectors of Y with eigenvalue 2μ. Indeed, for example, subtracting we have

$$Y(w_{11}-w_{22}) = 2\mu(w_{11}-w_{22}) \quad\text{and}\quad Y(w_{22}-w_{33}) = 2\mu(w_{22}-w_{33}). \tag{53}$$

This accounts for the diagonal strain tensors. Next let us compute the action of Y on the off-diagonal basis elements. For instance,

$$\begin{aligned} Y(w_{12}+w_{21})_{ij} &= [\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})]\,(\delta_{k1}\delta_{l2}+\delta_{k2}\delta_{l1})\\ &= \lambda\delta_{ij}\cdot 0+\mu\,[\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}+\delta_{i2}\delta_{j1}+\delta_{i1}\delta_{j2}]\,. \end{aligned} \tag{54}$$

Hence,

$$Y(w_{12}+w_{21}) = 2\mu(w_{12}+w_{21}), \quad Y(w_{23}+w_{32}) = 2\mu(w_{23}+w_{32}) \quad \& \quad Y(w_{31}+w_{13}) = 2\mu(w_{31}+w_{13}). \tag{55}$$

Thus we find that every symmetric off-diagonal matrix is an eigenvector of Y with eigenvalue 2μ. Combining, we find that any traceless symmetric matrix has eigenvalue 2μ. There are five such linearly independent matrices, say $w_{12}+w_{21}, w_{23}+w_{32}, w_{31}+w_{13}, w_{11}-w_{22}$ and $w_{22}-w_{33}$. Thus the 2μ-eigenspace has dimension five and consists of all shear tensors. Moreover, the corresponding eigenvalue is just twice the shear modulus. Exercise: Write out Y as a 6×6 matrix in the above basis, find its characteristic polynomial and thence its eigenvalues.

1.7 Elastic force density & Navier-Cauchy equations of elastostatics

Let us compute the force acting on a surface enclosing a finite volume V due to the material lying outside by integrating the stress over the surface ∂V. The i^{th} component of the force is

$$F_i = -\int_{\partial V} T_{ij}\, n_j\, dA. \tag{56}$$

Here dA is a small surface element on ∂V with normal vector n_j. Using Gauss' divergence theorem we can convert the surface integral to a volume integral to get

$$\mathbf{F} = -\int_V (\nabla \cdot \mathbf{T})\, dV \tag{57}$$

Since the above expression holds for an arbitrary volume, we may express the elastic force density $\mathbf{f}$ in terms of the divergence of the stress tensor:

$$\mathbf{f} = -\nabla \cdot \mathbf{T}. \tag{58}$$

For an isotropic medium $\mathbf{f}$ takes a simple form in terms of the divergence and Laplacian of the displacement field. Indeed, recall that the stress tensor for such a medium is given by (33). Taking its divergence we get the elastic force density

$$f_i = -\partial_j T_{ij} = (\lambda + \mu)\, \partial_i(\partial_j \xi_j) + \mu \partial_j^2 \xi_i = \left(K + \frac{\mu}{3}\right)\, \partial_i(\partial_j \xi_j) + \mu \partial_j^2 \xi_i. \tag{59}$$

The condition for elastostatic equilibrium in the presence of gravity (represented through the acceleration due to gravity $\mathbf{g}$) is

$$\mathbf{f} + \rho \mathbf{g} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{T} = \rho \mathbf{g}. \tag{60}$$

Here the elastostatic force density for an isotropic material is given by

$$\mathbf{f} = \left(K + \frac{\mu}{3}\right) \nabla(\nabla \cdot \xi) + \mu \nabla^2 \xi. \tag{61}$$

Thus, we arrive at the Navier-Cauchy equation for elastostatic equilibrium of an isotropic material

$$\left(K + \frac{\mu}{3}\right) \nabla(\nabla \cdot \xi) + \mu \nabla^2 \xi + \rho \mathbf{g} = 0. \tag{62}$$

This is a second order PDE for the displacement field ξ. It may be simplified in the absence of external forces.

1.8 Harmonic and bi-harmonic equations for expansion and displacement fields

• Ignoring body forces in the NC equation, the elastic force density must vanish in elastostatic equilibrium:

$$\mathbf{f} = -\nabla \cdot \mathbf{T} = \left(K + \frac{\mu}{3}\right) \nabla(\nabla \cdot \xi) + \mu \nabla^2 \xi = 0. \tag{63}$$

Re-expressing $K = E/(3(1-2\nu))$ and $\mu = E/(2(1+\nu))$ in terms of Young's modulus E and Poisson's ratio ν we get

$$\nabla(\nabla \cdot \xi) + (1-2\nu)\nabla^2 \xi = 0. \tag{64}$$

Now writing $\nabla^2 \xi = \nabla(\nabla \cdot \xi) - \nabla \times (\nabla \times \xi)$ we get

$$2(1-\nu)\nabla(\nabla \cdot \xi) - (1-2\nu)\nabla \times (\nabla \times \xi) = 0. \tag{65}$$

Taking the divergence of this equation and identifying the expansion $\Theta = \nabla \cdot \xi$ we get

$$\nabla^2 \Theta = 0. \tag{66}$$

• To obtain the bi-harmonic equation for ξ, we take the Laplacian of (64) which eliminates the first term since $\nabla^2 \Theta = 0$ to get

$$\nabla^2 \nabla^2 \xi = 0. \tag{67}$$

1.9 Energy of deformation and Elastic potential energy

• The elastic potential energy density is the work done per unit volume, in distorting an elastic material so that it acquires the strain tensor e_{ij}. We will see that it is given by $u = \frac{1}{2} Y_{ijkl} e_{ij} e_{kl}$ and that the total elastic potential energy stored in the material is $U = \int u \, d^3\mathbf{r}$. This formula shows that Y can be taken symmetric under $ij \leftrightarrow kl$. In other words, Y is a symmetric operator on the 6-dimensional linear space of symmetric strain tensors e. In elasto-static equilibrium, the strains in the body must be such that the elastic potential energy is an extremum.

• Recall from thermodynamics, that in general, the work done to take a system from an initial to final state depends on the process and not just the initial and final states. This is due to the possibility of heat transfer. However, we shall assume that the work done in deforming the solid is done slowly (reversibly and quasi-statically) with no heat exchange in such a way that the solid is in approximate elastostatic equilibrium at all intermediate times. This ensures that the work done depends only on the final state (defined by a displacement field ξ and strain e) as the initial state is assumed to be an undeformed solid. Now we may build up the displacement field ξ over, say, a unit time, in a particularly convenient way, by supposing the instantaneous displacement field is given by $\xi'(\mathbf{r}, t) = t\xi(\mathbf{r})$ for $0 \le t \le 1$.

• To compute the work done in deforming an elastic solid, it is convenient to have a formula for the work done by a force that is linear in the displacement, as elastic forces are of this sort.

• Thus, consider first the work done in elongating a wire by a length ξ within the range of validity of Hooke's law. The work done by the external force, (which must be equal and opposite to the restoring force by the assumption of quasi-staticity) $F = k\xi'$ which is proportional to the extension may be expressed in a convenient way

$$W = \int_0^\xi F(\xi') d\xi' = \int_0^\xi k\xi' d\xi' = \frac{1}{2} k\xi^2 = \frac{1}{2} F(\xi)\xi. \tag{68}$$

• More generally, if a force $F_i(\vec{\xi'})$ linear in the vectorial displacement ξ' produces a displacement that goes from 0 to $\vec{\xi}$ along the curve $\vec{\xi'}(t) = t\vec{\xi}$ for $0 \le t \le 1$, then the work done by the force is given by $W = F_i(\xi)\xi_i/2$. To see this, we take $F_i(\xi') = k_{ij}\xi'_j$ so that

$$W = \int_0^\xi F_i(\xi') d\xi'_i = \int_0^1 k_{ij} t \xi_j \xi_i dt = k_{ij}\xi_i\xi_j \frac{1}{2} = \frac{1}{2} F_i(\xi)\xi_i. \tag{69}$$

This formula will be useful to us in what follows, we can apply it to any force that is linear in displacement, even if the force and displacement are vectors.

• To compute the work done in deforming an elastic body of general shape, we break it up into small cubes and add up the work done in deforming each cube.

• Thus consider a small cube of side L centered at $(0,0,0)$ such that the faces are normal to x, y and z axes of the coordinate system. The work done in deforming this cube is the sum of the work done by surface and body forces.

• The i^{th} component of the force exerted by the external agent on the surface whose normal points along $\hat{x}$ is $-T_{i1}L^2$. Suppose the displacement produced by this force is $\vec{\xi}$, then by applying (69) the work done by this surface force is $-\xi_i(L/2)T_{i1}(L/2)L^2/2$. Similarly the work done by the external agent on the left face is $\xi_i(-L/2)T_{i1}(-L/2)L^2/2$. To add these, it is convenient to express each of these as a Taylor expansion around $x = 0$ for small $L/2$. Thus the sum of the work done on the left and right faces is

$$W_L + W_R = \frac{1}{2} L^2 \left(\xi_i(0) T_{i1}(0) + \frac{\partial (T_{i1}\xi_i)}{\partial x_1} \left(-\frac{L}{2} \right) - \xi_i(0) T_{i1}(0) - \frac{\partial (T_{i1}\xi_i)}{\partial x_1} \frac{L}{2} \right). \qquad (70)$$

Evidently the lowest order terms cancel. Identifying L^3 as the volume of the cube we get

$$\frac{1}{L^2}(W_L + W_R) = -\frac{1}{2}\partial_1 (T_{i1}\xi_i) \qquad (71)$$

The contributions of the top, bottom, fore and aft faces are

$$\frac{1}{L^2}(W_T + W_B) = -\frac{1}{2}\partial_3 (T_{i3}\xi_i) \quad \text{and} \frac{1}{L^2}(W_F + W_A) = -\frac{1}{2}\partial_2 (T_{i2}\xi_i). \qquad (72)$$

Adding the contributions of all faces, the total work done per unit volume by surface forces is

$$w_s = -\frac{1}{2} \sum_{i,j} \frac{\partial (T_{ij}\xi_i)}{\partial x_j}. \qquad (73)$$

To compute the work done by body forces such as gravity we note that by the condition for elastostatic equilibrium (60) we have $\mathbf{f}^b = -\mathbf{f} = \nabla \cdot \mathbf{T}$. Again using (69), the work done per unit volume by body forces is $w_b = \frac{1}{2}(\nabla \cdot T)_i \xi_i = \frac{1}{2}\xi_i \partial_j T_{ij}$. Adding the work done by surface and body forces we get the total work done per unit volume, which is the energy density

$$u = w_s + w_b = -\frac{1}{2}\partial_j(\xi_i T_{ij}) + \frac{1}{2}\xi_i \partial_j T_{ij} = -\frac{1}{2}(\partial_j \xi_i) T_{ij} = -\frac{1}{2} S_{ij} T_{ij} = -\frac{1}{2} e_{ij} T_{ij}. \qquad (74)$$

We replaced S by its symmetric part e since only the symmetric part can contribute on contraction with the symmetric tensor T_{ij}.

• Adding up the work done on all the elementary cubes, the elastic potential energy stored in a medium is given by

$$U = -\frac{1}{2} \int T_{ij} e_{ij} \, d^3\mathbf{r} = \frac{1}{2} \int Y_{ijkl} e_{ij} e_{kl} d^3\, \mathbf{r}. \qquad (75)$$

From this expression it is clear that only the part of Y_{ijkl} that is symmetric under the exchange $ij \leftrightarrow kl$ contributes to U.

• Interestingly, the tensorial form of Hooke's law can be regarded as the statement that the stress tensor is the negative gradient of the elastic potential energy U regarded as a function of the strain e: Indeed

$$T_{ij} = -\frac{\delta U}{\delta e_{ij}} = -Y_{ijkl} e_{kl}. \qquad (76)$$

• **Elastic potential energy of an isotropic material:** For a homogeneous isotropic medium the elastic potential energy can be expressed in terms of two elastic moduli

$$\begin{aligned} U[e] &= \frac{1}{2} \int Y_{ijkl} e_{ij} e_{kl} \, d^3r = \frac{1}{2} \int \left[\lambda \delta_{ij} \delta_{kl} e_{ij} e_{kl} + \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) e_{ij} e_{kl} \right] d^3r \\ &= \frac{1}{2} \int \left[\lambda (\text{ tr } \mathbf{e})^2 + 2\mu \text{ tr } \mathbf{e}^2 \right] \, d^3r. \end{aligned} \qquad (77)$$

This may also be expressed as

$$U[e] = \int \left[\frac{1}{2}K\Theta^2 + \mu\Sigma_{ij}\Sigma_{ij}\right] d^3r \tag{78}$$

by using the decomposition $e_{ij} = \Sigma_{ij} + \frac{1}{3}\Theta\delta_{ij}$ which allows us to write

$$e_{ij}e_{ij} = \Sigma_{ij}\Sigma_{ij} + \frac{1}{3}\Theta^2 \quad \text{as} \quad \Sigma_{ii} = 0. \tag{79}$$

1.10 Tensor of elasticity and potential energy for a cubic crystal

A crystal is an anisotropic material and we would expect the tensor of elasticity to have more than two linearly independent components. Let us consider an infinite cubic crystal (simple cubic or primitive cubic, e.g. Pyrite FeS_2). A cubic crystal has a discrete group of translational symmetries $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ generated by shifts by the lattice spacing along each of the crystal axes, taken along the x, y and z axes. This is a discrete subgroup of the group of translation symmetries $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ of a homogeneous material. If the crystal is large compared to the lattice spacing and we are only interested in phenomena on a scale large compared to the spacing, then we have approximate homogeneity, so Y_{ijkl} may be taken independent of position. A priori, the tensor of elasticity Y_{ijkl} has 81 components of which we have already argued at most 21 can be independent. We expect the discrete symmetries of the crystal to relate various components Y_{ijkl} to each other. Suppose we situate ourselves at a crystal vertex. A cubic crystal is symmetric under rotations by right angles about the three lattice vectors, which we take along $\hat{x}, \hat{y}, \hat{z}$. A rotation by 2π is the identity. So the rotational symmetries about the x axis form the cyclic group C_4 as do the rotational symmetries about the y and z axes. Each of these cyclic groups C_4 is a discrete subgroup of the corresponding SO(2) rotation symmetry group of an isotropic material about the various axes. We may compose such rotations about possibly different axes to get more general rotational symmetries of the cubic crystal. In addition, the crystal is symmetric under reflections in the three crystal xy, yz, zx-planes, which form the group $C_2\times C_2\times C_2$. Physical properties of the crystal must be unchanged under these symmetry transformations.

• The components of the tensor of elasticity may be segregated into those having 4 same indices, like Y_{xxxx}, those having precisely three identical indices, like Y_{xyyy}, those with precisely two repeated indices, like Y_{xxyz}, and those with two pairs of distinct repeated indices, like Y_{xxyy}. Let us see what the symmetries imply for all these components.

• Invariance by right angle rotations imply that Young's moduli for elongation in the three cardinal directions must be equal. Now $T_{xx} = -Y_{xxxx}e_{xx} + \cdots$ while $T_{yy} = -Y_{yyyy}e_{yy} + \cdots$ etc. Suppose we set up an elongation in the x direction with e_{xx} alone non-zero, it would result in a certain T_{xx}. If instead, we had a deformation with e_{yy} alone non-zero and having the same value, then the resulting T_{yy} must be the same as the previously determined T_{xx}. Thus we must have $Y_{xxxx} = Y_{yyyy} = Y_{zzzz}$.

• We can show using reflection symmetry that all Y_{ijkl}, where an index appears an odd number of times (e.g. Y_{xyyz} or Y_{xyyy}) must vanish.

• Recall that the elastic potential energy density $u = \frac{1}{2}Y_{ijkl}e_{ij}e_{kl}$ must be reflection invariant. Consider for example the term $Y_{xyyy}e_{xy}e_{yy}$. Under a reflection in the yz plane, $e_{xy} \to -e_{xy}$

while e_{yy} is unchanged. For the energy to remain unchanged, Y_{xyyy} must go to $-Y_{xyyy}$. But the reflected crystal has the same properties as the original one, so Y_{xyyy} must equal $-Y_{xyyy}$, so it must be zero. Similarly, all components with precisely one lone index (e.g. Y_{yzyy}, there are $3 \times 2 \times 4 = 24$ of these) or precisely two lone indices (e.g. Y_{xyzz}, there are $3 \times \binom{4}{2} \times 2 = 36$ of these) must be zero. Thus 60 of the 81 components must vanish.

• The remaining 18 components have two distinct indices each repeated twice, like Y_{xxyy} and Y_{xyxy}. But by 90 degree rotation + reflection invariance we conclude that replacement of all x's by y's and vice versa should not change the value of elastic constants. Thus $Y_{xxyy} = Y_{yyxx} = Y_{zzxx} = Y_{zzyy} = Y_{yyzz}$ etc. and $Y_{xyxy} = Y_{zyzy} = Y_{zxzx}$ etc.

• Combining, we have only 3 classes of non-zero components, $Y_{xxxx} = Y_{yyyy} = Y_{zzzz}$,

$$Y_{xxyy} = Y_{yyxx} = Y_{zzxx} = Y_{zzyy} = Y_{yyzz} = \ldots, \quad \text{and} \quad Y_{xyxy} = Y_{zyzy} = Y_{zxzx} = \ldots. \tag{80}$$

Thus for a cubic crystal, the tensor of elasticity has three independent components, to be contrasted with the two independent components for a homogeneous isotropic solid.

1.11 Bending of a beam - Cantilever bridge

Out treatment is based on the discussion in Blandford and Thorn §11.5.

Here we consider a beam of rectangular cross section with length l, width w and height (thickness) t clamped rigidly at one end and extending horizontally in the absence of gravity. We work with a coordinate system attached to the beam with $\hat{x}$ along the length of the beam, $\hat{y}$ along the width and $\hat{z} = \hat{x} \times \hat{y}$ ($\hat{z}$ is vertically upwards when the beam is horizontal). The origin is located at center of the clamped end. When the beam bends under its weight, this coordinate system is not cartesian. In a curvilinear coordinate system, we need to use covariant rather than partial derivatives in evaluating the strain tensor or the divergence of the stress tensor etc. However, for small deformations, the difference can be shown to be of higher order in small quantities, so we do not worry about this issue here.

• When the beam is allowed to sag due to gravity each elemental volume is displaced by $\vec{\xi}(x,z)$ (we have translation invariance in the y direction along the width of the beam). The upper part of the beam stretches while the lower part is compressed. Thus, there must be a neutral surface on which the longitudinal strain $\xi_{x,x} = \partial_x \xi_x$ is zero. This neutral surface is of course curved downwards. x and y are the longitudinal and transverse coordinates on the neutral surface. We will suppose that the neutral surface is in the middle of the beam ($z = 0$) so that the top and bottom surfaces are at $z = \pm t/2$. Let the angle the neutral surface makes with the horizontal be $\theta(x)$ and its drop from the horizontal (through the undisturbed neutral surface) be $\eta(x) \geq 0$ (η always vanishes at the clamped end $x = 0$) (see Fig. 1.11). We wish to find the shape of the bent beam which is encoded in the profile $\eta(x)$ and thereby determine the sag $\eta(l)$ at the right extreme.

• Consider a portion of the beam extending from the longitudinal position x with length dx that is elongated by a small amount $\epsilon \ll dx$ measured at height z above the neutral surface. The arc length of this extended beam segment is given by $dx + \epsilon$ which subtends angle θ at the center of a circle of radius R where R is the radius of curvature. Consider the small right triangle in Fig.1.11 with small angle θ, adjacent side of length z and opposite side of length ϵ. This triangle is similar to the big right triangle with side opposite to angle θ having length

$dx + \epsilon$ and adjacent side of length R. Thus from the two similar triangles (see Fig.1.11)

$$\tan\theta = \frac{\epsilon}{z} = \frac{dx+\epsilon}{R}. \tag{81}$$

Thus the longitudinal strain (for $\epsilon \ll dx$) is given by

$$\xi_{x,x} = \frac{\epsilon}{dx} \approx \frac{\epsilon}{dx+\epsilon} = \frac{z}{R} = z\frac{d\theta}{dx}. \tag{82}$$

In the last equality, we have used the formula for curvature $\kappa = \frac{1}{R} = \frac{d\theta}{ds}$ where the infinitesimal arc length $ds \approx dx$. Now (from the 'horizontal' right triangle with adjacent and opposite sides dx and $d\eta$ to the angle θ) $\sin\theta \approx \theta = \frac{d\eta}{dx}$ for small angle θ and so we get

$$\frac{1}{R} = \eta''(x) \quad \text{and} \quad \xi_{x,x} = z\frac{d^2\eta}{dx^2}. \tag{83}$$

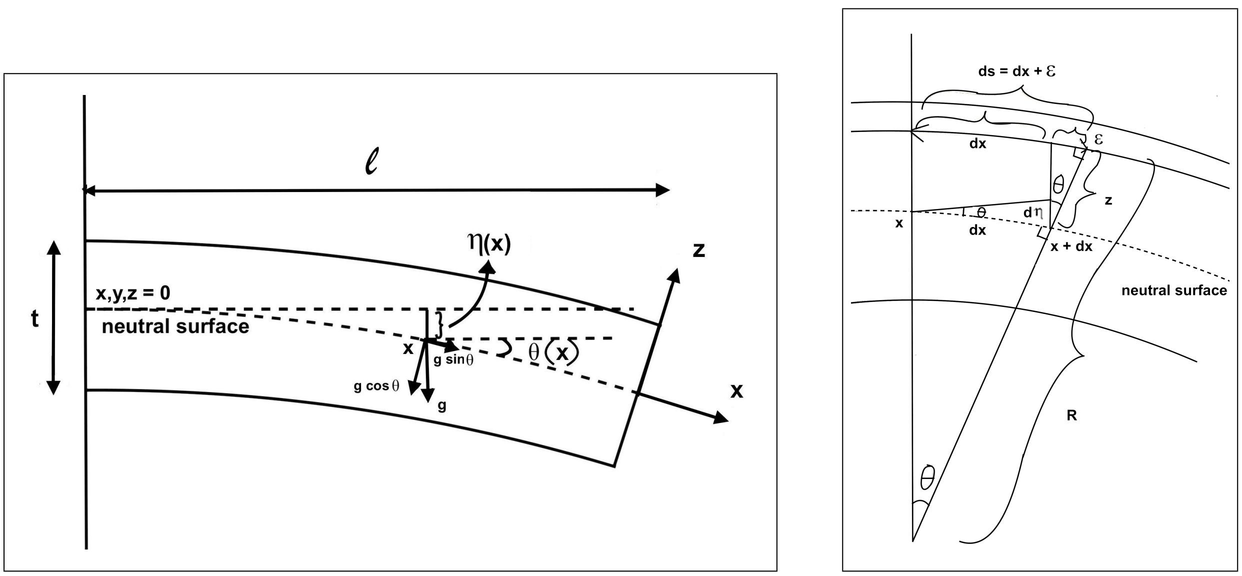

Figure 2: Bending of a cantilever bridge due to gravity: coordinates and similar triangles

• We may regard the beam as made of a bundle of long parallel fibres extending lengthwise to compute the longitudinal stress using Hooke's law:

$$T_{xx} = -E\xi_{x,x} = -Ez\frac{d^2\eta}{dx^2}. \tag{84}$$

Here, E is Young's modulus. By the Navier-Cauchy equation the horizontal component of the elastostatic force density must be balanced by the horizontal component of gravity in equilibrium. Using (58) and (60) it is given by

$$f_x = -T_{xi,i} = -T_{xx,x} - T_{xz,z} = Ez\frac{d^3\eta}{dx^3} - T_{xz,z} = -\rho g \sin\theta. \tag{85}$$

Due to translational invariance in y, $T_{xy,y} = 0$.

• Multiplying by z and integrating over the thickness $(-t/2 \le z \le t/2)$ of the beam we get

$$\frac{Et^3}{12}\frac{d^3\eta}{dx^3} = \int_{-t/2}^{t/2} z\partial_z T_{xz} dz + \rho g \sin\theta \int_{-t/2}^{t/2} z dz = [zT_{xz}]_{-t/2}^{t/2} - \int_{-t/2}^{t/2} T_{xz} dz = -\int_{-t/2}^{t/2} T_{xz} dz. \quad (86)$$

Here, the gravity integral vanishes while in the first term we integrated by parts and assumed that the shear stress vanishes on the upper and lower surfaces. This is true to the extent that we ignore the force of the air above and below the beam on the top and bottom layers of the beam - this is a good approximation for a heavy beam where air pressure is much smaller than elastic and gravitational forces.

• We may use this result to find a simple expression (in terms of η) for the 'vertical' shearing force $S(x)$ (normal to the neutral surface) which may be regarded as the restoring force exerted by an element at x on the element to its immediate right (see shear force Fig. 3). By definition, the vertical shearing force normal to the neutral surface is

$$S(x) = \int_{-t/2}^{t/2}\int_{-w/2}^{w/2} T_{zx} dz dy = w\int_{-t/2}^{t/2} T_{xz} dz = -\frac{Ewt^3}{12}\frac{d^3\eta}{dx^3} = -D\frac{d^3\eta}{dx^3}. \quad (87)$$

• Here we used the symmetry $T_{xz} = T_{zx}$. The proportionality factor

$$D = E\int_{-t/2}^{t/2}\int_{-w/2}^{w/2} z^2 dz dy = Ew\frac{t^3}{12} \quad (88)$$

is called the flexural rigidity.

• We are interested in balancing the torque on any segment of the beam. The part of the beam above the neutral surface has been expanded due to the bending and wishes to contract while the lower part is compressed and wishes to expand. The combination of these two tendencies is to produce a counterclockwise 'bending torque' $M(x)$ due to the bending of the beam. The $\hat{y}$ component of this *bending torque* due to the force $T_{xx} dy dz$ exerted by an element A on its neighbour B to the right is given by (see Fig. 4)

$$M(x) = \sum \mathbf{r}\times\mathbf{F} = \int zT_{xx} dy dz. \quad (89)$$

Here we have added up the torques (all pointing along $\hat{y}$) computed about the points $(x, y, z = 0)$, (holding x fixed) for each longitudinal section of the beam by integrating over all values of y along its width.

• Since there is no longitudinal stress at the right extreme of the beam (no force on air), $T_{xx}(x = l) = 0$ so that $M(l) = 0$. We may express $M(x)$ in terms of η by using (84) and the expression for flexural rigidity (88). Indeed, we have

$$M(x) = -E\eta''(x)\iint z^2 dy dz = -D\eta''(x) \quad \text{or} \quad M = -\frac{D}{R}. \quad (90)$$

It follows that $\eta''(l) = 0$. Moreover, we expect $\eta'' \ge 0$ due to the convex shape of the bent beam as seen from above. Thus $M \le 0$ and contributes a net counterclockwise torque.

• From (87) we also deduce that the vertical shear force is the derivative of the bending torque

$$S(x) = \frac{dM}{dx} \quad \text{or} \quad S dx = dM. \quad (91)$$

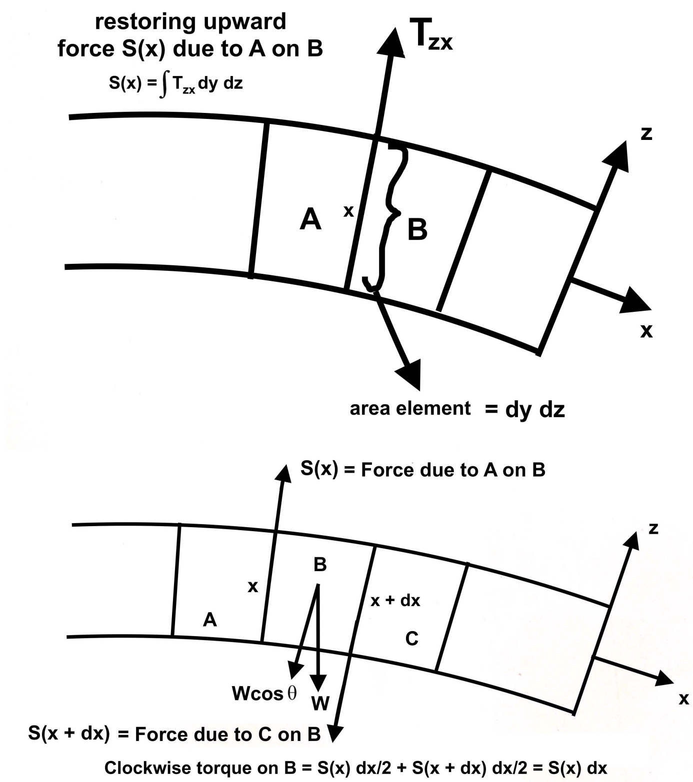

Figure 3: Vertical shearing force, and its torque.

This may be regarded as a torque balance equation where Sdx is the torque due to the vertical restoring stress T_{zx} (see Fig. 3) and dM is the bending torque due to the horizontal stress T_{xx}.

• In equilibrium the z ('upward') component of the Navier-Cauchy force balance equation must also be satisfied:

$$f_z - \rho g \cos\theta = 0 \quad \text{or} \quad - T_{zx,x} - T_{zz,z} - \rho g \cos\theta = 0. \tag{92}$$

Integrating over thickness and width of the beam we get

$$\int_{-t/2}^{t/2} \int_{-w/2}^{w/2} (T_{zx,x} + T_{zz,z}) dz dy = -\rho g w t \cos\theta = -W \cos\theta, \tag{93}$$

where W is the weight per unit length. From (87) we get

$$\frac{dS}{dx} + T_{zz}(z = t/2) - T_{zz}(z = -t/2) = -W \cos\theta \tag{94}$$

As before, assuming the stress vanishes on the upper and lower surfaces of the beam we get

$$\frac{dS}{dx} = -W \cos\theta. \tag{95}$$

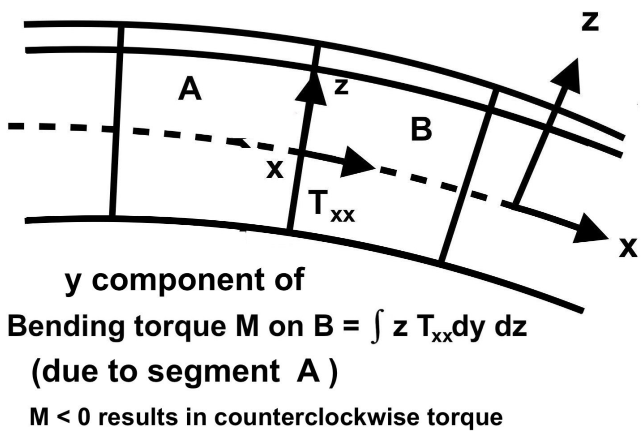

Figure 4: Bending torque

Using (87) and approximating $\cos\theta \approx 1$ for small θ we get a fourth order differential equation for the vertical displacement field $\eta(x)$:

$$\frac{d^4\eta}{dx^4} = \frac{W}{D}. \tag{96}$$

We may solve this fourth order differential equation using four appropriate boundary conditions. Since the beam is clamped at $x = 0$ we have $\eta(0) = \eta'(0) = 0$. On the other hand, the bending torque ($M \propto \eta''$) and shear force ($S \propto \eta'''$) vanish at the free end giving $\eta''(l) = \eta'''(l) = 0$.

• Thus we get

$$\eta(x) = \frac{W}{D}\left(\frac{x^4}{24} - \frac{lx^3}{6} + \frac{l^2x^2}{4}\right) \quad \text{for} \quad 0 \le x \le l. \tag{97}$$

Thus the right extreme of the beam drops down a height $\eta(l) = \frac{Wl^4}{8D}$. Notice that $\eta''(x) = (W/2D)(x-l)^2 > 0$.

• For a beam of fixed length, the deflection $\eta(l)$ is inversely proportional to the flexural rigidity D. A simple illustration of the effect of this scaling may be found in joists that support floors. Joists are several long cuboid shaped wooden beams going between walls and on top of which the upper floor rests. According to Blandford and Thorne, joists are typically $w = 2''$ wide and $t = 6''$ tall ('tall and thin') and of much greater length and may be treated as cantilevers supported by walls at either end. However, if the joists are flipped so that they are short and wide, i.e., of height $t' = 2''$ and $w' = 6''$ then the flexural rigidity reduces by a factor of $D/D' = wt^3/w't'^3 = 2 \times 6^3/(6 \times 2^3) = 9$ and the floor may be expected to sag 9 times as much!

1.12 Buckling bifurcation

• See also §11.6 of Blandford and Thorne and §38-5 of Feynman Lectures Vol 2.

• Consider a playing card or visiting card of small thickness t width w and length l ($t \ll w \lesssim l$) held with its length horizontal. It is subject to compressional forces F acting parallel to its length on the left and right edges of width w which are held fixed (see Fig. 1.12).

• Empirically we find that for small compressive stresses, the card remains with its face horizontal and unbent. The card of course contracts lengthwise by $\Delta l \approx Fl/Ewt$. However, for larger forces, we find that the horizontal flat face configuration is unstable to buckling. The card bends or buckles as shown in the figure. It may buckle either upwards or downwards with roughly equal probability. Since the card is light, gravity plays a negligible role. The main forces acting on the card are the external forces F and the internal stresses T_{ij}. Euler was perhaps the first to study the buckling of a beam quantitatively and he found the critical minimal force F_{crit} for buckling to take place.

• Let us try to understand this buckling by a simple analysis of the forces and torques acting on the card. We suppose that the card has a flexural rigidity $D = Ewt^3/12$. Since the card is very thin, we expect it to have a smaller D and be more susceptible to bending than a beam of the same material with greater thickness but comparable width.

• Unlike for in the cantilever, we use Cartesian coordinates with x along the undisturbed length of the card and z the vertical transverse coordinate while the card is translation invariant along its width which is in the y direction.

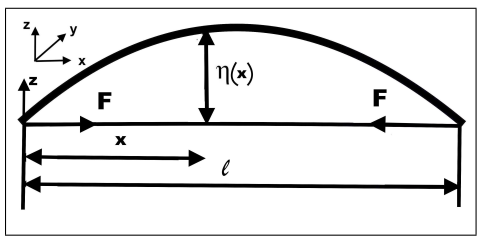

Figure 5: Lengthwise cross section of buckled card subjected to compressive force F.

• Let us consider the portion of the card that extends from the horizontal coordinate x to the right extreme $x = l$. The total torque about the point $(x, y, z = \eta(x))$ (for each fixed y along the width of the card) acting on this portion of the card must vanish. It is the sum of the bending torque $M(x)\hat{y}$ due to the stress exerted by the section of the card to its immediate left and the torque due to the external force at $x = l$. The torque due to the external force F is $F\eta\hat{y}$. On the other hand, the bending torque $M\hat{y}$ may be read off from Eq. (90) of §1.11. Since here η is measured from below the card while in the cantilever it was measured from above, we have $M(x) = D\eta''(x)$ where D is the flexural rigidity. We will neglect the small differences that arise from measuring x horizontally rather than along the card. Balancing the torques we get

$$(M + F\eta)\hat{y} = 0 \quad \text{or} \quad D\frac{d^2\eta}{dx^2} + F\eta = 0. \tag{98}$$

Clearly, an unbent card $\eta \equiv 0$ is a solution for any force F. Remarkably, there are forces for which there are other solutions as well. Indeed, if the force is unspecified, we may view this as an eigenvalue problem for $\eta(x)$ with eigenvalue $-F/D$. The solutions satisfying the boundary conditions $\eta(0) = \eta(l) = 0$ (here we ignore the fact that the ends of the card move in slightly from their undisturbed locations $x = 0$ and $x = l$) are

$$\eta_n(x) = A_n \sin k_n x \quad \text{with} \quad k_n = \left(\frac{F}{D}\right)^{1/2} = \frac{n\pi}{l} \quad \text{for } n = 0, 1, 2, 3 \ldots. \tag{99}$$

Therefore, there is a critical 'Euler' force

$$F_{\text{critical}} = \frac{\pi^2 D}{l} \tag{100}$$

below which there is no solution other than $\eta \equiv 0$ corresponding to $n = 0$ (an unbent card). At $F = F_{\text{critical}}$ the unbent card is still a solution but there is another solution corresponding to $n = 1$ which looks like an arched card. This is an example of a bifurcation where there is more than one solution of the equations for the same external conditions. Here the unbent card is the unstable solution while the buckled cards are the stable ones. In fact, within this linear approximation, the amplitude of A_1 of the buckled card is not determined and can be arbitrary (but small).

• With a card it is difficult to realize the higher modes $n \geq 2$. However, these may be seen in mountain folding. When tectonic plates are compressed in one direction, mountains (such as the Jura mountains in France or even the Himalayas) can form by *folding*. These roughly parallel rows of mountain chains may be regarded as a higher mode in the above model.

• This system gives us a simple example of **spontaneous symmetry breaking**. The original card and the forces F are symmetric under up-down reflection. So we may expect to find the card in a state that is up-down symmetric even in the presence of the force. However, this is not the case for sufficiently large F. We say that the up-down symmetry is spontaneously broken by the bent state in which the card is found. However, the symmetry is not altogether lost. Rather, the symmetry transformation permutes states in which the card may be found. Indeed, application of the vertical reflection converts an upward bent card to a card bent downwards. It of course takes a flat card to itself. When the equations of a system possess a symmetry that is not manifested in its state of lowest energy, we say that the symmetry is spontaneously broken.

• **Effective potential for buckling bifurcation:** Even though the card is a system with infinitely many degrees of freedom we may try to model the above bifurcation by focussing on just one coordinate η_0 which is the maximum deflection of the card. We will model the system using an effective potential or free energy $V(\eta_0)$ whose extrema represent the equilibrium configurations. The minima of V should correspond to stable card configurations while its maxima should correspond to unstable configurations. Let us enumerate some desirable features that this potential could possess.

1. Since the card could equally well bend upwards or downwards, we wants the set of extrema of V to be closed under reversal of sign. If V is even then it will guarantee that its extrema come in pairs $\pm\eta_0^*$.

2. $V(\eta_0)$ should depend parametrically on the external force F.

3. For $F < F_{\text{crit}}$, V must have only one extremum. It should be a minimum at $\eta_0 = 0$ corresponding to an unbent card. As F grows, the minimum should get flatter as its stability gets weakened.

4. For $F > F_{\text{crit}}$, V must have 3 extrema: a local maximum at $\eta_0 = 0$ (as the flat card is unstable to small perturbations) and a pair of global minima at $\eta_0 = \pm\eta_0^*$ with $V(\eta_0^*) = V(-\eta_0^*)$, as the card bent in either direction is stable to small perturbations.

The simplest effective potential with these properties is a quartic (bi-quadratic) function of η_0, $V(\eta_0) = \lambda(F)(\eta_0^2 - (\eta_0^*(F))^2)^2$ where η_0^* is a function of F with the property that $|\eta_0^*| > 0$ for $F > F_{\text{critical}}$ and zero for $F \leq F_{\text{critical}}$. To model the bifurcation we observe that as F increases, $V(\eta_0)$ must become become flatter at $\eta_0 = 0$. For $F > F_{\text{critical}}$, the minimum at $\eta_0 = 0$ becomes a maximum surrounded by two minima on either side (a double well potential). The maximum at $\eta_0 = 0$ is unstable while the minima on either side ($\pm\eta_0^*$) are stable equilibria (see Fig 6). These two minima of V correspond to a card that has buckled either upwards or downwards relative to the flat card.

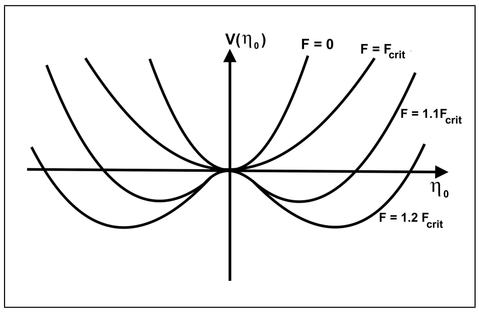

Figure 6: Effective potential of a card that buckles as a function of the maximum deflection η_0 for various values of external force F.

2 Elastodynamics

2.1 Equations of elastodynamics

• In an elastic body in stable static equilibrium, the stresses (both internal and external) and strains adjust themselves so that Hooke's law $T_{ij} = -Y_{ijkl}e_{kl}$ is satisfied, the elastic potential energy is a minimum and the Navier-Cauchy equation is satisfied. But out of equilibrium, when there are unbalanced external or internal forces, parts of the material may move a bit, so that the displacement field $\xi(\mathbf{r}, t)$ is a function of time. We are interested for example in the situation resulting from strains due to the application of external stresses on a body, which are subsequently withdrawn. Restoring stresses should develop resulting in deformation and relative motion of parts of the body. We assume the motions are small displacements about equilibrium locations, so ξ is small, and Hooke's law should still relate the stress and strain tensors at any instant of time. Here $\mathbf{r}$ labels material elements by specifying their equilibrium locations. So $\xi(\mathbf{r}, t)$ is a Lagrangian variable since it refers to the displacement of a particular material element. The instantaneous location of this material element is $\mathbf{r} + \xi(\mathbf{r}, t)$. Its instantaneous velocity is $\frac{\partial \xi}{\partial t}$ and its acceleration is $\frac{\partial^2 \xi}{\partial t^2}$. Suppose its mass is $dm = \rho dV$ where dV is its volume, which is an elemental volume located around $\mathbf{r} + \xi$. ρ therefore is the density at the location $\mathbf{r} + \xi$. Now dV could change and move around as the element vibrates, and ρ could also change, but dm remains fixed by conservation of mass.

The mass $\times$ acceleration of this element is $\rho dV \frac{\partial^2 \xi}{\partial t^2}$. Here ξ is assumed small. Any departure of $\rho(\mathbf{r} + \xi)$ from $\rho(\mathbf{r})$ would be of order ξ. So if we work to lowest order in infinitesimals, we may evaluate ρ at $\mathbf{r}$ in the expression $\rho\, dV\, \frac{\partial^2 \xi}{\partial t^2}$. This mass $\times$ acceleration must equal the force acting on the element whose undisturbed location is $\mathbf{r}$. This force is usually comprised of two parts, (a) external body forces that can be described as a force per unit volume $\mathbf{f}_{body}$ (e.g. gravity $\mathbf{f}_{body} = \rho\mathbf{g}$ where $\mathbf{g}$ is the acceleration due to gravity) and (b) surface forces that act across area elements in the material and are due to neighboring material. If we ignore surface forces, then Newton's force equation per unit volume says that $\rho\ddot{\xi} = \rho\mathbf{g}$. To find the surface force on the small element that is centered at $\mathbf{r}+\xi$, we recall that the small force acting across a small surface $\hat{n}dA$ is $\mathbf{F}(\hat{n}dA, \mathbf{r}+\xi)$, due to the material on the side from which $\hat{n}$ points on the material on the side to which $\hat{n}$ points. We may approximate $\mathbf{F}(\hat{n}dA, \mathbf{r}+\xi) \approx \mathbf{F}(\hat{n}dA, \mathbf{r})$ since the difference is second order in infinitesimals. Moreover, by the definition of the stress tensor, $\mathbf{F}_i = T_{ij}\hat{n}_j dA$. So let $\hat{n}$ denote the outward pointing normal to the surface ∂dV enclosing an element contained in volume dV. Then the i^{th} component of the surface force acting on this element may be expressed as a volume integral using the divergence theorem

$$i^{\text{th}}\text{component of surface force } = -\int_{\partial dV} T_{ij} n_j dA = -\int_{dV} \partial_j T_{ij} dV. \tag{101}$$

Using Hooke's law $T_{ij} = -Y_{ijkl}e_{kl}$ Newton's equation for ξ becomes

$$\rho\ddot{\xi}_i = \rho\mathbf{g}_i + \partial_j Y_{ijkl} e_{kl}. \tag{102}$$

• The case of a homogeneous isotropic material is particularly interesting. In this case, the surface force on an element occupying small volume dV is (61)

$$f_i dV = -\partial_j T_{ij} dV = \left[(\lambda+\mu)\partial_i \nabla\cdot\xi + \mu\nabla^2\xi_i\right] dV. \tag{103}$$

This plus the body force must equal $\rho dV\ddot{\xi}$. Canceling out dV, the equation of motion of linear elastodynamics for a homogeneous isotropic material is $(\lambda+\mu = K+\mu/3)$

$$\rho\frac{\partial^2\xi}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\xi) + \mu\nabla^2\xi + \rho\mathbf{g}. \tag{104}$$

If the material is isotropic but inhomogeneous, λ, μ would (in general) be functions of location $\mathbf{r}$ and the equations would involve derivatives of λ, μ as well. The difference between $\rho(\mathbf{r}+\xi) \approx \rho(\mathbf{r}) + \xi\cdot\nabla\rho(\mathbf{r})$ and $\rho(\mathbf{r})$ is first order in the infinitesimal displacements ξ. So to first order in infinitesimals we may take $\rho(\mathbf{r}+\xi) \approx \rho(\mathbf{r})$ in the acceleration term. Aside from the inhomogeneous body force, which is like a source, this is a homogeneous linear PDE, second order in both space and time derivatives of the displacement field ξ.

2.2 Material derivative

• In the Eulerian description, we are interested in the time development of various variables like velocity, pressure, density and temperature at a given location (point of observation) $\vec{r} = (x,y,z)$ in the material. The change in say, density, at a fixed location is $\frac{\partial\rho(\vec{r})}{\partial t}$. However, different material particles will arrive at the point $\vec{r}$ as time elapses. It is also of interest to know how the corresponding dynamical variables evolve, not at a fixed location but for a fixed small material element, as in a Lagrangian description[3]. This is especially important since the dynamical laws

[3] By a small material element, we mean a collection of molecules that is sufficiently numerous so that concepts such as 'volume occupied by the element' make sense and yet small by macroscopic standards so that the velocity, density e.t.c., are roughly constant over its extent. For instance, we may divide a container with 10^{23} molecules into 10000 fluid elements, each containing 10^{19} molecules.

of mechanics apply directly to the material particles, not to the point of observation. So we may ask how a variable changes along the 'flow', so that the observer is always attached to a fixed material element. For instance, the change in density of a fixed element in a small time dt as it moves from location $\mathbf{r}$ to the displaced location $\mathbf{r}+d\mathbf{r}$ is

$$d\rho = \rho(\mathbf{r}+d\mathbf{r}, t+dt) - \rho(\mathbf{r},t) \approx d\mathbf{r}\cdot\nabla\rho + \frac{\partial\rho}{\partial t}\,dt \tag{105}$$

We divide by dt and take the limit $dt \to 0$ and observe that $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the material. Thus the instantaneous rate of change of density of a material element that is located at $\mathbf{r}$ at time t is

$$\frac{D\rho}{Dt} \equiv \frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho = (\partial_t + v_x\partial_x + v_y\partial_y + v_z\partial_z)\,\rho \tag{106}$$

$\frac{D}{Dt} = \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$ is called the material (sometimes total, substantial or convective or fisherman's) derivative. It is related to the Lie derivative along a vector field. It can be used to express the rate of change of a physical quantity (velocity, pressure, temperature etc.) associated to a fixed fluid element, i.e., along the flow specified by the velocity field $\vec{v}$. This formula for the material derivative bears a resemblance to the rigid body formula relating the time derivatives of a vector (e.g. momentum or angular momentum) relative to the lab and co-rotating frames: $\left(\frac{d\mathbf{A}}{dt}\right)_{lab} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{co}-\text{mov}} + \mathbf{\Omega}\times\mathbf{A}$, where Ω is the angular velocity of the rigid body. A quantity f (could be a scalar or a vector) is said to be conserved along the flow or dragged by the flow if its material derivative vanishes $\frac{Df}{Dt} = 0$.

• Similarly, the material derivative of velocity $\frac{D\mathbf{v}}{Dt} = \partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$ gives us the instantaneous acceleration of the material element that is at $\mathbf{r}$ at the time t.

• Since $\frac{D}{Dt}$ is a first order partial differential operator, Leibnitz's product rule of differentiation holds $\frac{Dfg}{Dt} = f\frac{Dg}{Dt} + \frac{Df}{Dt}g$ for scalar functions f, g. Similarly for a scalar f and vector field $\vec{w}$, we check that the Leibnitz rule holds

$$\frac{D(f\vec{w})}{Dt} = \frac{Df}{Dt}\,\vec{w} + f\frac{D\vec{w}}{Dt}. \tag{107}$$

2.3 Conservation of mass and momentum in elastodynamics

In this section we derive the equations of elastodynamics for the displacement field ξ from an Eulerian viewpoint (by contrast with the Lagrangian description of §2.1) using conservation of mass and momentum. When linearized, the resulting equations reduce to those of §2.1.

• **Conservation of mass:** Consider material of mass density $\rho(\mathbf{x},t)$ occupying a region. Now consider a small sub-volume V fixed in space. Let us assume that there are no sources or sinks of material in V. Suppose $\mathbf{v}(\mathbf{r},t)$ is the instantaneous Eulerian velocity of the material that occupies the location $\mathbf{r}$ at time t. Then the flux i.e., rate at which the mass moves across a unit area on the boundary ∂V of V is $\rho\mathbf{v}\cdot\hat{n}$ where $\hat{n}$ is the outward pointing normal. Mass conservation then requires

$$\frac{\partial}{\partial t}\int \rho dV = -\int_{\partial V}\rho\mathbf{v}\cdot d\Sigma \quad \text{or} \quad \frac{\partial}{\partial t}\int \rho dV = -\int_V \nabla\cdot(\rho\mathbf{v})dV \tag{108}$$

by Gauss' divergence theorem. Here, $d\vec{\Sigma}$ is the elemental outward-pointing surface area vector on the boundary ∂V. Since this must be true for an arbitrary volume dV we get the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{109}$$

We say that mass is locally conserved and that the mass density ρ and mass current $\rho\mathbf{v}$ satisfy a local conservation law, which is also called the continuity equation.

• In elasticity, it is often convenient to work with the displacement $\xi(\mathbf{x}, t)$ rather than velocity field $\mathbf{v}(\mathbf{x}, t)$. $\mathbf{v}(\mathbf{r}, t)$ is the instantaneous velocity of the material at the location $\mathbf{r}$, thus it is given by the partial derivative $\mathbf{v} = \frac{\partial \xi}{\partial t}$ of the instantaneous displacement of the material at $\mathbf{r}$.

• **Conservation of momentum:** Each component of momentum density ρv_i, just like mass density ρ is locally conserved. The i^{th} component of momentum dp_i crossing an area element $d\vec{\Sigma}$ per unit time is given by $\rho v_i \mathbf{v} \cdot d\vec{\Sigma} = \rho v_i v_j d\Sigma_j$. This is a vector linear in the vector $d\vec{\Sigma}$. By 'peeling off' $d\Sigma_j$, we may define the second rank mechanical momentum flux density (or current) tensor T^m_{ij}:

$$dp_i = T^m_{ij} d\Sigma_j \quad \text{where} \quad T^m_{ij} = \rho v_i v_j. \tag{110}$$

In more detail, T_{ij} is the i^{th} component of the momentum crossing a unit surface in the direction of its normal which is assumed to point in the j^{th} direction. Consequently, $T_{ij} n_j$ is the i^{th} component of the momentum crossing a unit surface in the direction of its normal $\hat{\mathbf{n}}$. The mechanical momentum flux tensor $\mathbf{T}^m_{ij}$ is manifestly symmetric and quadratic in velocities. In the absence of momentum sources or sinks we have

$$\partial_t \int_V \rho v_i \, dV = -\int_{\partial V} T_{ij} \, d\Sigma_j = -\int_V \partial_j T_{ij} \, dV. \tag{111}$$

As this is true for any fixed volume V we obtain a local conservation law for momentum:

$$\partial_t(\rho \mathbf{v}_i) + \partial_j \mathbf{T}^m_{ij} = 0 \quad \text{or} \quad \partial_t(\rho \mathbf{v}) + \nabla \cdot \mathbf{T}^m = 0. \tag{112}$$

This equation ignores momentum transport due to elastic forces. More generally, when an elastic medium is deformed, there is an elastic restoring force per unit volume $\mathbf{f}(\mathbf{r}) = -\nabla \cdot \mathbf{T}^{\text{el}}$ on the material element at $\mathbf{r}$. Thus in an elastic medium the local conservation of momentum takes the form

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\mathbf{T}^{\text{el}} + \mathbf{T}^m) = 0. \tag{113}$$

By Hooke's law the elastic stress tensor is given in terms of the displacement field by $T^{\text{el}}_{ij} = -Y_{ijkl} e_{kl}$ so that this may be viewed as a PDE for the density and displacement fields along with the continuity equation. Notice that the equation is quadratically non-linear in velocities through the mechanical stress tensor.

• **Equation of elastodynamics:** We may combine mass and momentum conservation to obtain a first order (in time) evolution equation for the velocity field $\mathbf{v}(r, t)$. When expressed in terms of the displacement field ξ this becomes a second order non-linear evolution equation called the equation of elastodynamics. It is obtained by multiplying the continuity equation $\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$ by $\mathbf{v}$ and subtracting it from the momentum conservation equation

$$\rho_t v_i + v_i \partial_j (\rho v_j) = 0 \quad \text{and} \quad v_i \rho_t + \rho \frac{\partial v_i}{\partial t} + \partial_j (\rho v_i v_j) + \partial_j T^{\text{el}}_{ij} = 0 \tag{114}$$

to get

$$\rho\left[\frac{\partial v_i}{\partial t}+v_j\partial_j v_i\right]=-\partial_j T^{\rm el}_{ij}\quad\text{or}\quad \rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})=-\nabla\cdot T^{\rm el}. \tag{115}$$

This is a quadratically non-linear equation for $\mathbf{v}=\frac{\partial\xi}{\partial t}$. Including external body forces and expressing the LHS in terms of the material derivative we get

$$\rho\frac{D\mathbf{v}}{Dt}=-\nabla\cdot T^{\rm el}+\mathbf{f}_{body}. \tag{116}$$

Recalling that the material derivative of the Eulerian velocity $\mathbf{v}$ is the acceleration of the material element at $\mathbf{r}$ at time t, we see that this is Newton's second law per unit volume.

• For typical elastic deformations, the non-linear advection term may be ignored. To see why, suppose L and T are characteristic length and time scales associated to the system. Then the ratio of the second term $\rho\mathbf{v}\cdot\nabla\mathbf{v}$ to the first term $\rho\partial_t\mathbf{v}$ is of the order the strain (ξ/L) where ξ is the magnitude of the displacement field. For typical materials the strain $O(\xi/L)\leq 10^{-3}$. Thus we may ignore the second term to get a linearized equation for the displacement field ξ:

$$\rho\frac{\partial^2\xi}{\partial t^2}=-\nabla\cdot\mathbf{T}_{\rm el}. \tag{117}$$

For a homogeneous isotropic medium with constant elastic moduli this becomes

$$\rho\frac{\partial^2\xi}{\partial t^2}=(\lambda+\mu)\nabla(\nabla\cdot\xi)+\mu\nabla^2\xi. \tag{118}$$

This equation (which coincides with (104) upon inclusion of body forces) is to be solved in conjunction with the continuity equation. However, for most elastic deformations, ρ may be treated constant. This is because the fractional change in density has the same magnitude as the expansion Θ which is small in many circumstances. To see this, consider a small material volume of fixed mass M and density $\rho=M/V$. If the mass density ρ changes by a small amount $\delta\rho$ producing a change δv in the specific volume $v=1/\rho$ (volume per unit mass) then

$$v+\delta v=\frac{1}{\rho+\delta\rho}\quad\text{or}\quad v\left(1+\frac{\delta v}{v}\right)=\frac{1}{\rho}\left(\frac{1}{1+\frac{\delta\rho}{\rho}}\right)\approx\frac{1}{\rho}\left(1-\frac{\delta\rho}{\rho}\right). \tag{119}$$

Since the mass of the material element does not change, $\frac{\delta v}{v}=\frac{\Delta V}{V}=\nabla\cdot\xi=\Theta$, we get

$$\frac{\delta\rho}{\rho}=-\nabla\cdot\xi=-\Theta\quad\text{with}\quad|\Theta|\lesssim 10^{-3}. \tag{120}$$

2.4 Comparison between elastodynamic and electromagnetic wave equations

The equation of motion of elastodynamics (104) bears a resemblance to the vector wave equation in electrodynamics, for the vector potential. To see this, we begin with Maxwell's equations (in cgs un-rationalized Heaviside-Lorentz units)

$$\begin{aligned}\nabla\cdot\mathbf{B}=0,&\qquad \nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\cdot\mathbf{E}=\rho\quad\text{and}&\quad \nabla\times\mathbf{B}=\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}.\end{aligned} \tag{121}$$

The homogeneous equations on the first line state the absence of magnetic monopoles, and Faraday's law of induction. The inhomogeneous equations on the second line are Gauss' law and Ampere's law with Maxwell's correction term involving the time derivative of the electric field (the displacement current). The electric charge and current densities must in addition satisfy the continuity equation $(1/c)\partial_t \rho + \nabla \cdot \mathbf{j} = 0$.

• The first pair of homogeneous Maxwell equations are identically satisfied if the fields are expressed in terms of scalar and vector potentials $(\phi, \mathbf{A})$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{122}$$

However, the gauge potentials $(\phi, \mathbf{A})$ are not uniquely determined by the $\mathbf{E}$ and $\mathbf{B}$ fields, more on this momentarily. In terms of the gauge potentials, the Ampere-Maxwell equation becomes (use $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$)

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mathbf{j} - \frac{1}{c}\partial_t \nabla\phi - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{123}$$

This equation bears some resemblance to the equation of elastodynamics for a homogeneous isotropic material. The body force per unit volume $\rho\mathbf{g} = \mathbf{f}$ plays the role of the current density. Though there is no elastic analogue for the scalar potential term, $\mathbf{A}$ plays the role of ξ. However, while ξ is the directly measurable displacement field, $\mathbf{A}$ is not uniquely determined by the measurable electric and magnetic fields. Two gauge potentials $(\phi, \mathbf{A})$ and $(\phi', \mathbf{A}')$ which differ by a gauge transformation (here $\chi(\mathbf{r}, t)$ is an arbitrary scalar function)

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \quad \phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}. \tag{124}$$

correspond to the same electromagnetic fields. Gauge transformations form a group $\mathcal{G}$ which acts on the space of gauge potentials $\mathcal{A} = \{(\phi, \mathbf{A})\}$. Each orbit (equivalence class of gauge potentials) corresponds to an electromagnetic field $(\mathbf{E}, \mathbf{B})$ and the space of electromagnetic fields is the quotient $\mathcal{A}/\mathcal{G}$. A choice of orbit representatives is called a gauge choice. It is obtained by imposing condition(s) on the gauge potentials which are satisfied by one set of gauge potentials from each equivalence class.

• A convenient gauge choice is Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Given a vector potential $\mathbf{A}'$ we find its representative in Coulomb gauge by making the gauge transformation $\mathbf{A} = \mathbf{A}' - \nabla\chi$ with χ chosen to satisfy Poisson's equation $\nabla^2\chi = \nabla \cdot \mathbf{A}'$.

• Gauss' law simplifies in Coulomb gauge: $\nabla \cdot E = -\nabla^2\phi - \frac{1}{c}\frac{\partial \nabla \cdot \mathbf{A}}{\partial t} = 0$ becomes $-\nabla^2\phi = \rho$, whose solution involves the Coulomb potential (this is why $\nabla \cdot A = 0$ is called the Coulomb gauge!) $\phi(\mathbf{r}, t) = \frac{1}{4\pi}\int d^3r' \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}$. In particular, in Coulomb gauge, the scalar potential $\phi(\mathbf{r}, t)$ is not a dynamical quantity, it is entirely fixed by the instantaneous charge density. Now let us specialize to the case where there are no charges present in the interior and boundary of the region of interest, so that $\rho = 0$. Then $\phi = 0$. In the absence of charges, Coulomb gauge is called radiation gauge ($\phi = 0$, $\nabla \cdot \mathbf{A} = 0$), since electromagnetic radiation is most easily described in this gauge. In radiation gauge, the Ampere-Maxwell equation becomes

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla^2 \mathbf{A} + \mathbf{j}, \qquad (\text{provided } \nabla \cdot \mathbf{A} = 0, \phi = \rho = 0). \tag{125}$$

This is the vector wave equation in the presence of a current source $\mathbf{j}$. One is often interested in EM waves in vacuum, in which case $\mathbf{j} = 0$ and we get the homogeneous vector wave equation.

• The Coulomb/radiation gauge condition $\nabla \cdot \mathbf{A}$ is often called the transversality condition. To see why, let us consider a monochromatic plane wave that propagates along the wave vector $\mathbf{k}$

$$\mathbf{A}(\mathbf{r}, t) = \tilde{\mathbf{A}}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{126}$$

The direction of the Fourier amplitude $\tilde{\mathbf{A}}_{\mathbf{k}}$ of this mode is called the direction of polarization. Check that $\nabla \cdot \mathbf{A} = 0$ implies $\mathbf{k} \cdot \tilde{\mathbf{A}}_{\mathbf{k}} = 0$. This means the polarization must be orthogonal to the direction of propagation. Thus there can be only two linearly independent (transverse) propagating components of the vector potential. These correspond to the two independent polarizations of electromagnetic radiation. For the above plane wave to satisfy the vector wave equation, ω and $\mathbf{k}$ must satisfy the dispersion relation $\omega^2 = c^2 \mathbf{k}^2$. Since the wave equation is linear, both the real and imaginary parts of this plane wave are solutions and we may restrict to them when seeking real physical solutions.

• The equations of elastodynamics

$$\rho \frac{\partial^2 \xi}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \xi) + \mu \nabla^2 \xi + \mathbf{f}^{\mathbf{b}} \tag{127}$$

reduce to the vector wave equation if $\lambda + \mu = 0$. Comparing, the speed of transverse elastic waves should be $c = \sqrt{\mu/\rho}$. λ, μ are both non-negative for typical materials, so we expect waves in elastic media to be a superposition of transverse and longitudinal disturbances, as we will see below.

2.5 Compressional and shear waves in a homogeneous isotropic elastic medium

• Here we investigate solutions of the equations of elastodynamics, especially in unbounded media in the absence of body forces. We use the Helmholtz decomposition theorem[4] of vector calculus to write the displacement field as a sum of a divergence-free and a curl-free part

$$\xi = \xi_T + \xi_L \quad \text{where} \quad \nabla \cdot \xi_T = 0 \quad \text{and} \quad \nabla \times \xi_L = 0. \tag{128}$$

We will show that ξ_T and ξ_L each satisfies a vector wave equation (assuming the density variations $\nabla \rho$ may be ignored). ξ_T being divergence-free is a transverse wave ($\mathbf{k} \cdot \tilde{\xi}_T = 0$ in Fourier space). It does not cause any change in volume or density, it is called a shear wave. On the other hand ξ_L is not divergence free and describes a longitudinal compressional wave (a sound wave). These two waves have distinct characteristic speeds of propagation. To begin with, the equation of elastodynamics is

$$\rho(\ddot{\xi}_T + \ddot{\xi}_L) = (\lambda + \mu) \nabla (\nabla \cdot \xi_L) + \mu \nabla^2 (\xi_T + \xi_L). \tag{129}$$

We may try to eliminate ξ_T by taking the divergence of this equation and using $\nabla \cdot \xi_T = 0$ and $\nabla \rho = 0$. We get

$$\nabla \cdot (\rho \ddot{\xi}_L) = (\lambda + \mu) \nabla^2 \nabla \cdot \xi_L + \mu \nabla^2 \nabla \cdot \xi_L = (\lambda + 2\mu) \nabla \cdot (\nabla^2 \xi_L) \tag{130}$$

[4] Helmholtz's theorem says that a smooth vector field in $\mathbb{R}^3$ that vanishes faster than $1/r$ at infinity can be uniquely decomposed as a sum of divergence-free and curl-free parts.

So we have

$$\nabla \cdot \left(\rho\ddot{\xi}_L - (\lambda + 2\mu)\nabla^2 \xi_L\right) = 0. \tag{131}$$

Thus we have

$$\nabla \cdot \left(\rho\ddot{\xi}_L - (\lambda + 2\mu)\nabla^2 \xi_L\right) = 0 \quad \text{and} \quad \nabla \times \left(\rho\ddot{\xi}_L - (\lambda + 2\mu)\nabla^2 \xi_L\right) = 0. \tag{132}$$

The second equation follows since ξ_L is curl-free. Thus the vector field in parentheses is both curl free and divergence-free. By Helmholtz's theorem on the uniqueness of vector fields with specified divergence and curl, it must be the zero vector field (assuming it vanishes sufficiently fast at infinity). Thus ξ_L satisfies a vector wave equation

$$\rho\frac{\partial^2 \xi_L}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \xi_L \tag{133}$$

It describes curl-free 'sound' waves of expansion/compression, since $\nabla \cdot \xi_L \neq 0$. These waves propagate at the speed $c_{\text{sound}} = \sqrt{\frac{\lambda+2\mu}{\rho}}$.

• On the other hand, we may try to eliminate ξ_L by taking the curl of the equation of elastodynamics assuming ρ to be constant. We get

$$\nabla \times (\rho\ddot{\xi}_T - \mu\nabla^2 \xi_T) = 0 \quad \text{and} \quad \nabla \cdot (\rho\ddot{\xi}_T - \mu\nabla^2 \xi_T) = 0 \tag{134}$$

The second equation follows from $\nabla \cdot \xi_T = 0$. Thus the vector field in parentheses is both curl free and divergence-free and must be identically zero by Helmholtz's uniqueness theorem. Hence ξ_T satisfies a vector wave equation

$$\rho\ddot{\xi}_T = \mu\nabla^2 \xi_T \tag{135}$$

describing non-compressional waves ($\nabla \cdot \xi_T = 0$) that propagate with a speed $c_{\text{shear}} = \sqrt{\mu/\rho}$. These are called shear waves, they are a bit like electromagnetic waves in the sense that they are transverse waves, since $\nabla \cdot \xi_T = 0$. They travel at a lower speed than the sound waves described by ξ_L above.

• The ratio of the speeds of sound and shear waves is independent of the density of the medium

$$\frac{c_{\text{sound}}}{c_{\text{shear}}} = \sqrt{\frac{2\mu + \lambda}{\mu}} \tag{136}$$

Measurement of this ratio in seismic waves during an earth quake can give information about the elastic constants of the region (of the Earth's crust) through which the waves propagated.

2.6 Plane wave solutions for shear and compressional waves

We seek monochromatic plane wave solutions $\xi(\mathbf{x}, t) \propto \xi_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ (for constant density) to the linear elastodynamic wave equation (127) without body forces. The substitutions $\partial_t \to -i\omega, \nabla^2 \to -k^2, \nabla \to i\mathbf{k}, \nabla\cdot \to i\mathbf{k}\cdot$ and $\nabla\times \to i\mathbf{k}\times$ turn the PDE into a system of decoupled algebraic equations for the Fourier modes $\xi_{\mathbf{k}}$:

$$\rho\omega^2 \xi_{\mathbf{k}} = \mu k^2 \xi_{\mathbf{k}} + (\lambda + \mu)\,\mathbf{k}(\mathbf{k}\cdot\xi_{\mathbf{k}}) = \mu k^2 \xi_{\mathbf{k}} + \left(K + \frac{1}{3}\mu\right)\mathbf{k}(\mathbf{k}\cdot\xi_{\mathbf{k}}) \tag{137}$$

Using the Helmholtz decomposition and monochromatic, plane wave ansatz for the displacement field ξ we see that ξ_L is the longitudinal component of the oscillation along $\hat{k}$ and ξ_T is the transverse component of the oscillation perpendicular to $\hat{k}$:

$$\nabla \times \xi_L = 0 \quad \text{and} \quad \nabla \cdot \xi_T = 0 \quad \Rightarrow \quad \mathbf{k} \times \tilde{\xi}_L = 0 \quad \text{and} \quad \tilde{\xi}_T \cdot \hat{k} = 0 \tag{138}$$

where $K = \lambda + 2\mu/3$. Longitudinal waves have only one degree of freedom $\tilde{\xi}_L = \tilde{\xi}_L \hat{k}$ and are completely describable by the single polarization component $\tilde{\xi}_L$. On the other hand, $\tilde{\xi}_T$ can have two independent polarizations since it can point in any of the two directions perpendicular to the direction of propagation $\hat{k}$. Thus an elastodynamic wave, unlike a EM wave in vacuum can have a total of three possible polarizations. Interestingly, even EM waves can have a longitudinal polarization component when light travels through a plasma or a superconductor.

• The dispersion relations for longitudinal and transverse waves follow from the algebraic equation (137)

$$\omega_L^2 = (\lambda + 2\mu) k^2/\rho = \left(K + \frac{4}{3}\mu\right) k^2/\rho \quad \text{and} \quad \omega_T^2 = \mu k^2/\rho. \tag{139}$$

2.7 Energy of elastodynamic waves in an isotropic homogeneous medium

Just like electromagnetic waves or waves on a string, elastodynamic waves also transport energy. The energy density of such waves in an isotropic, homogenous medium includes a kinetic contribution $(\rho \dot{\xi}^2)/2$ in addition to the elastic potential energy of Eq. (78) leading to the total energy

$$E = \int \left(\frac{1}{2} \rho \dot{\xi}^2 + \frac{1}{2} K \Theta^2 + \mu \Sigma_{ij} \Sigma_{ij} \right) d\mathbf{r}. \tag{140}$$

Let us derive the analogue of this formula for energy stored in the longitudinal waves and compute it in a polychromatic example. Dotting (133) with $\dot{\xi}_L$ and integrating we get

$$\int \left[\rho \dot{\xi}_L \cdot \frac{\partial^2 \xi_L}{\partial t^2} - (\lambda + 2\mu) \dot{\xi}_L \cdot \nabla^2 \xi_L \right] d\mathbf{r} = \int \left[\frac{d}{dt} \left(\frac{\rho \dot{\xi}_L^2}{2} \right) + (\lambda + 2\mu)(\partial_j \dot{\xi}_{Li})(\partial_j \xi_{Li}) \right] d\mathbf{r} = 0$$

$$\text{or} \quad \frac{d}{dt} \int \frac{1}{2} \left(\rho \dot{\xi}_L^2 + (\lambda + 2\mu)(\partial_j \xi_{Li})(\partial_j \xi_{Li}) \right) d\mathbf{r} = \frac{d}{dt} \int \frac{1}{2} \left(\rho \dot{\xi}_L^2 + (\lambda + 2\mu) S_{ij}^L S_{ij}^L \right) d\mathbf{r} = 0 \tag{141}$$

upon integrating by parts.

• To calculate the potential energy we take $\hat{k}$ along $\hat{z}$ so that the longitudinal displacement may be written as a Fourier integral:

$$\vec{\xi}_L(z) = \xi_{L,z} \hat{z} = \int \left[\tilde{\xi}_{L,k} e^{i(kz - \omega_L(k)t)} + \tilde{\xi}_{L,k}^* e^{-i(kz - \omega_L(k)t)} \right] [dk] \; \hat{z}. \tag{142}$$

Here $[dk] = dk/2\pi$ and $\omega_L(k) = |k|\sqrt{(K + 4\mu/3)/\rho}$. The second term is the complex conjugate of the first and ensures that $\vec{\xi}_L(z)$ is real. Moreover the first term includes both right- and left-moving waves as k takes both positive and negative values though $\omega_L(k) \geq 0$ by definition. It follows that $\partial_x \xi_x = \partial_y \xi_y = 0$ so that the expansion becomes $\Theta = \nabla \cdot \xi = \partial_z \xi_z = S_{zz}$. All other components of S_{ij} vanish. Thus, the total energy of the longitudinal wave becomes

$$E_L = \int \frac{1}{2} \left(\rho \dot{\xi}_L^2 + (\lambda + 2\mu) \Theta^2 \right) d\mathbf{r} = \int \frac{1}{2} \left(\rho \dot{\xi}_L^2 + \left(K + \frac{4}{3}\mu \right) \Theta^2 \right) d\mathbf{r}. \tag{143}$$

We compute the KE and PE by going to Fourier space

$$
\begin{aligned}
KE_L &= \frac{\rho}{2}\int[dk_1dk_2](-i\omega_L(k_1))(-i\omega_L(k_2))\tilde{\xi}_{L,k_1}\tilde{\xi}_{L,k_2}\int dze^{i(k_1+k_2)z}e^{-i(\omega_L(k_1)+\omega_L(k_2))t}\int dxdy \\
&\quad+\frac{\rho}{2}\int[dk_1dk_2](-i\omega_L(k_1))(i\omega_L(k_2))\tilde{\xi}_{L,k_1}\tilde{\xi}^*_{L,k_2}\int dze^{i(k_1-k_2)z}e^{-i(\omega_L(k_1)-\omega_L(k_2))t}\int dxdy \\
&\quad+\frac{\rho}{2}\int[dk_1dk_2](i\omega_L(k_1))(i\omega_L(k_2))\tilde{\xi}^*_{L,k_1}\tilde{\xi}^*_{L,k_2}\int dze^{-i(k_1+k_2)z}e^{i(\omega_L(k_1)+\omega_L(k_2))t}\int dxdy \\
&= \frac{\rho A}{2}\int[dk]\omega_L(k)^2\left[2|\tilde{\xi}_{L,k}|^2-\tilde{\xi}_{L,k}\tilde{\xi}_{L,-k}e^{-2i\omega_L(k)t}-\tilde{\xi}^*_{L,k}\tilde{\xi}^*_{L,-k}e^{2i\omega_L(k)t}\right]. \qquad (144)
\end{aligned}
$$

Despite appearances, KE_L is a positive quantity. Here $A=\int dxdy$ is the transverse area of the region and $\int dze^{i(k_1+k_2)z}=2\pi\delta(k_1+k_2)$. Similarly, we compute the potential energy by going to Fourier space,

$$
\begin{aligned}
PE_L &= \frac{A}{2}\left(K+\frac{4\mu}{3}\right)\int[dk_1dk_2](ik_1)(ik_2)\tilde{\xi}_{L,k_1}\tilde{\xi}_{L,k_2}\int dze^{i(k_1+k_2)z}e^{-i(\omega_L(k_1)+\omega_L(k_2))t} \\
&\quad+\frac{A}{2}\left(K+\frac{4\mu}{3}\right)\int[dk_1dk_2](-ik_1)(-ik_2)\tilde{\xi}^*_{L,k_1}\tilde{\xi}^*_{L,k_2}\int dze^{-i(k_1+k_2)z}e^{i(\omega_L(k_1)+\omega_L(k_2))t} \\
&\quad+A\left(K+\frac{4\mu}{3}\right)\int[dk_1dk_2](ik_1)(-ik_2)\tilde{\xi}_{L,k_1}\tilde{\xi}^*_{L,k_2}\int dze^{i(k_1-k_2)z}e^{-i(\omega_L(k_1)-\omega_L(k_2))t} \\
&= \frac{A}{2}\left(K+\frac{4\mu}{3}\right)\int[dk]k^2\left[2|\tilde{\xi}_{L,k}|^2+\tilde{\xi}_{L,k}\tilde{\xi}_{L,-k}e^{-2i\omega_L(k)t}+\tilde{\xi}^*_{L,k}\tilde{\xi}^*_{L,-k}e^{2i\omega_L(k)t}\right]. \qquad (145)
\end{aligned}
$$

Using the dispersion relation $\rho\omega_L^2=(K+\frac{4}{3}\mu)k^2$ (139) we see that the first terms in KE_L and PE_L are the same while the last two terms are equal in magnitude but opposite in sign. Though neither the KE_L nor PE_L is separately conserved the total energy is conserved and is given by the simple and manifestly time-independent formula

$$E_L=2A\left(K+\frac{4\mu}{3}\right)\int[dk]k^2|\tilde{\xi}_{L,k}|^2=2\rho A\int[dk]\omega_L(k)^2|\tilde{\xi}_{L,k}|^2. \qquad (146)$$

• For a monochromatic wave (only one wave number $k,-k$) ξ is periodic in time with period $T_k=2\pi/\omega_L(k)$ so that it is natural to average the PE and KE over the time period ($\langle f\rangle=\frac{1}{T}\int_0^T f dt$). Then the oscillatory terms in KE_L and PE_L average to zero and we see that the time-averaged KE_L and PE_L are equal:

$$E_L=2\langle KE_L\rangle=2\langle PE_L\rangle=2A\left(K+\frac{4\mu}{3}\right)\int[dk]k^2|\tilde{\xi}_{L,k}|^2. \qquad (147)$$

We say that on average the energy is equally partitioned between kinetic and potential energies.